# COEFFICIENTS OF DRINFELD MODULAR FORMS AND HECKE OPERATORS

*by*

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***Abstract.*** — Consider the space of Drinfeld modular forms of fixed weight and type for $\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(\mathbf{F}_q[T])$. It has a linear form $b_n$, given by the coefficient of $t^{m+n(q-1)}$ in the power series expansion of a type $m$ modular form at the cusp infinity, with respect to the uniformizer $t$. It also has an action of a Hecke algebra. Our aim is to study the Hecke module spanned by $b_1$. We give elements in the Hecke annihilator of $b_1$. Some of them are expected to be nontrivial and such a phenomenon does not occur for classical modular forms. Moreover, we show that the Hecke module considered is spanned by coefficients $b_n$, where $n$ runs through an infinite set of integers. As a consequence, for any Drinfeld Hecke eigenform, we can compute explicitly certain coefficients in terms of the eigenvalues. We give an application to coefficients of the Drinfeld Hecke eigenform $h$.

## 1. Introduction

Drinfeld modular forms are certain analogues over $\mathbf{F}_q[T]$ of classical modular forms, introduced by D. Goss [**12, 13**]. A Drinfeld modular form $f$ has a power series expansion with respect to a canonical uniformizer $t$ at the cusp infinity. If $f$ has type $m$, this expansion is $\sum_{n\geq 0} b_n(f)t^{m+n(q-1)}$. On the space of Drinfeld modular forms of fixed weight and type, we have the linear form $b_n : f \mapsto b_n(f)$ and an action of a Hecke algebra. In the present work, we investigate the Hecke module spanned by $b_1$.

Our interest in the problem comes from the torsion of rank-2 Drinfeld modules. In a previous work, we established a uniform bound on the torsion under an assumption on the latter Hecke module in weight 2 and type 1 (see [**1, 2**]). This condition was required for studying a Drinfeld modular curve at a neighborhood of the cusp infinity, namely for showing that the map from the curve (or rather a symmetric power) to a quotient of its Jacobian variety is a formal immersion at this cusp in a special fiber.

Before stating the main results, we fix some notations. Let $A = \mathbf{F}_q[T]$ be the ring of polynomials over a finite field $\mathbf{F}_q$ in an indeterminate $T$, $K = \mathbf{F}_q(T)$ the field of rational functions, $K_\infty = \mathbf{F}_q((1/T))$ and $\mathbf{C}_\infty$ the completion of an algebraic closure of $K_\infty$. For an ideal $\mathfrak{n}$ of $A$, $k \in \mathbf{N}$ and $0 \leq m < q-1$, we consider the $\mathbf{C}_\infty$-vector space $M_{k,m}(\Gamma_0(\mathfrak{n}))$ of Drinfeld modular forms of weight $k$ and type $m$ for the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $\mathrm{GL}_2(A)$ (see Section 4.1 for the definition). These are rigid analytic $\mathbf{C}_\infty$-valued functions on $\mathbf{C}_\infty - K_\infty$ which have an interpretation as multi-differentials on the Drinfeld modular curve attached to $\Gamma_0(\mathfrak{n})$.

Let $\mathbf{T} = \mathbf{T}_{k,m}(\Gamma_0(\mathfrak{n}))$ be the Hecke algebra, that is the commutative subring of $\mathrm{End}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})))$ spanned over $\mathbf{C}_\infty$ by all Hecke operators $T_P$ for $P$ monic polynomial in $A$ (see Section 4.2). Its restriction $\mathbf{T}' = \mathbf{T}'_{k,m}(\Gamma_0(\mathfrak{n}))$ to the subspace $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ of doubly cuspidal forms (with expansion vanishing at order $\geq 2$ at all cusps) stabilizes this subspace. As Goss first observed, doubly cuspidal Drinfeld modular forms play a role similar to classical cusp forms.

In this work, we are interested in the pairing between the space $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ and the Hecke algebra $\mathbf{T}'$ given by the coefficient $b_1$ of the expansion. More precisely, the dual space $\mathrm{Hom}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ has a natural right action of $\mathbf{T}$ (given by composition) and contains the linear form $b_n : f \mapsto b_n(f)$. Let $u = u_{k,m,\mathfrak{n}} : \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ be $\mathbf{C}_\infty$-linear map defined by $s \mapsto b_1 s$. Our main results concern the kernel $\mathbf{I}$ and the image $b_1\mathbf{T}'$ of $u$.

Let $A_{d+}$ be the set of monic polynomials of degree $d$ in $A$. The first statement gives a family of elements of $\mathbf{I}$.

***Theorem 1.1***. — *The following elements of* $\mathbf{T}'$ *belong to* $\mathbf{I}$*:*

1. $\sum_{P \in A_{1+}} P^{1-m} T_P + T_1$ *if* $m \in \{0, 1\}$.
2. $\sum_{P \in A_{d+}} C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}} T_P$ *if* $d \geq 1$ *and* $(i_0, \ldots, i_{d-1}) \in \mathbf{N}^d$ *is such that*

$$0 \leq i_j \leq q - m \qquad \text{for all } j \in \{0, \ldots, d-1\} \tag{1}$$

$$i_0 + \ldots + i_{d-1} \leq (d-1)(q-1) - m. \tag{2}$$

   *Here,* $C_{P,j} \in A$ *stands for the* $j$*th coefficient of the Carlitz module at* $P$ *(see Section 3.1 for its definition).*

3. $\sum_{P \in A_{d+}} P^l T_P$ *if* $0 \leq l \leq q - m$ *and* $d \geq 1 + (l+m)/(q-1)$
   $\sum_{P \in A_{d+}} T_P$ *if* $d \geq 2$*, or if* $d = 1$ *and* $m = 0$.

These elements actually belong to the span over $A$ of all Hecke operators. Moreover, they are universal in the sense that, for a given type $m$, they do not depend on the weight $k$ nor on the ideal $\mathfrak{n}$.

In most cases, we believe that $\mathbf{I} \neq 0$, that is at least one element of Theorem 1.1 is a nontrivial endomorphism of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$, hence the pairing is not perfect. Over the space $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ with $\mathfrak{n}$ prime, the situation is as follows. If $\mathfrak{n}$ has degree 3, we prove that $\mathbf{I} = 0$ (Theorem 7.7). If $\mathfrak{n}$ has degree $\geq 5$, numerical experiments suggest that $\mathbf{I} \neq 0$ (Conjecture 6.9). Moreover, it may happen that some elements of Theorem 1.1 are zero in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$: examples of such a situation are explored in Section 6.3.

For the rest of the introduction, we restrict our attention to Drinfeld modular forms of type 0 or 1. Our second statement gives an infinite family of coefficients of Drinfeld modular forms in $b_1\mathbf{T}'$.

***Theorem 1.2***. — *Assume* $q$ *is a prime and* $m \in \{0, 1\}$*. Let* $\mathscr{S}$ *be the set of natural integers of the form* $c/(q-1)$*, where* $c \in \mathbf{N}$ *is such that the sum of its base* $q$ *digits is* $q - 1$*. For every* $n \in \mathscr{S}$*, there exists* $s_n \in \mathbf{T}'$*, independent of* $k$ *and* $\mathfrak{n}$*, satisfying*

$$b_n = b_1 s_n \quad \in b_1\mathbf{T}'.$$

*Moreover, $b_1\mathbf{T}'$ is the $\mathbf{C}_\infty$-vector space spanned by $b_n$ for all $n \in \mathscr{S}$.*

The primality assumption on $q$ is not essential (see Remark 7.3). As for the set $\mathscr{S}$, it is infinite of natural density zero and the first integer not belonging to $\mathscr{S}$ is $q+1$. For example, if $q = 3$, the first elements of $\mathscr{S}$ are

$$1, 2, 3, 5, 6, 9, 14, 15, 18, 27, 41, 42, 45, 54, 81.$$

Theorem 1.2 relies on an explicit version, Theorem 7.2 (the elements $s_n$ that we produce depend on whether the type is 0 or 1). The expression for $s_n$ is rather natural: it is a $A$-linear combination of Hecke operators $T_P$, with $P$ of fixed degree, involving Carlitz binomial coefficients in $A$.

Suppose now that $\mathbf{I} \neq 0$. Then the map $u$ fails to be surjective (see Lemma 6.2). In particular, $b_1\mathbf{T}'$ does not contain all linear forms $b_n$ for $n \geq 1$. It is then natural to ask what is the smallest integer $n$ such that $b_n \notin b_1\mathbf{T}'$. Theorem 1.2 suggests that $n = q+1$ might be a good candidate.

Both theorems bring new insight on Drinfeld Hecke eigenforms. Consider a Drinfeld modular form $f$ which is an eigenform for the Hecke algebra $\mathbf{T}$. Theorem 1.1 translates into linear relations among eigenvalues of $f$, provided that $b_n(f) \neq 0$ for some $n \in \mathscr{S}$ (Proposition 6.5 and Corollary 7.5). Similarly, Theorem 7.2 gives explicit formulas for coefficients $b_n(f)$ ($n \in \mathscr{S}$) in terms of eigenvalues of $f$ and $b_1(f)$. From Theorem 7.2, we also derive:

- multiplicity one statements in some spaces of Drinfeld modular forms of small dimension (Theorem 7.7); as far as we know, these are the only known results of this kind for Drinfeld modular forms.
- explicit expressions for some coefficients of the Drinfeld modular form $h$ (Proposition 8.1). This extends previous work of Gekeler.

As a side remark, we give a brief account of the multiplicity one problem for Drinfeld modular forms. Since there exist two Hecke eigenforms for $\mathrm{GL}_2(A)$ with different weights and same system of eigenvalues (Goss [**12**]), the question of multiplicity one should be stated as: do eigenvalues and weight determine the Hecke eigenform, up to a multiplicative constant? (see Gekeler [**7**], Section 7). Böckle and Pink showed that this does not hold for doubly cuspidal forms of weight 5 for the group $\Gamma_1(T)$ when $q > 2$ by means of cohomological techniques (Example 15.4 of [**3**]). Except for Theorem 7.7 mentioned above, the question remains open for $\Gamma_0(\mathfrak{n})$.

We now compare our results with their analogues for classical modular forms. Consider the space $S_k(\Gamma_0(N))$ of cuspidal modular forms of weight $k$ for the subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$ ($N \geq 1$). Let $(c_n(f))_{n\geq 1}$ be the Fourier coefficients of such a modular form $f$ at the cusp infinity. Computing the action of the $n$th Hecke operator $T_n$ on the Fourier expansion of $f$ gives the well-known relation, for any $n \geq 1$

$$c_n(f) = c_1(T_n f). \tag{3}$$

In particular, the Hecke module spanned by the linear form $c_1$, which now contains all coefficients $c_n$, is the whole dual space of $S_k(\Gamma_0(N))$ and the coefficient $c_1$ gives rise to a perfect pairing over $\mathbf{C}$ between $S_k(\Gamma_0(N))$ and the Hecke algebra. Conjecture 6.9 and Theorem 1.2 thus suggest a phenomenon specific to the function field setting. For Drinfeld modular forms, the reason for not having straightforward statements about the kernel and image of $u$ is that the action of Hecke operators on the expansion is not well understood. Goss [**12, 13, 11**] and subsequently Gekeler [**7**] wrote down this action using Goss polynomials. But such polynomials are difficult to handle (see also Remark 5.3). In particular, a relation as general as (3) is lacking.

We now sketch the proofs of Theorems 1.1 and 1.2, which involve rather elementary techniques.

- We first compute the coefficient $b_1(T_P f)$, for any $f$ and $P$, using Goss polynomials (Proposition 5.5). Note that the formula we get is more intricate than (3): it is a $A$-linear combination of several coefficients of $f$. For the next step, the crucial point is that the index of these coefficients depends only on the degree of $P$. This already proves that $b_1\mathbf{T}'$ is contained in the $\mathbf{C}_\infty$-vector space spanned by $b_n$, for $n \in \mathscr{S}$ when $m \in \{0, 1\}$ (Corollary 5.8).
- We take advantage of characteristic $p$. For power sums of polynomials of a given degree in $A$, vanishing properties and closed formulas are well-known (see [**21**, III] for a survey). Here we use a variant consisting of power sums of coefficients of the Carlitz module. Such sums are studied in Section 3 and closed formulas are given in Proposition 3.5. In Section 3.4, we also explain their connection with Carlitz binomial coefficients and special values of Goss zeta function at negative integers.
- By taking adequate linear combinations of $b_1(T_P f)$, for $P$ of fixed degree, and using results of Section 3, we obtain elements in the kernel $\mathbf{I}$ (Theorem 1.1, Section 6) and in the image $b_1\mathbf{T}'$ (Theorems 7.2 and 1.2).

For the study of the Hecke module $b_1\mathbf{T}'$, our method has reached its limit and improving our results would require new ideas. Our approach might be used to tackle other Hecke modules $b_i\mathbf{T}'$: however, computing $b_i(T_P f)$ for any $i \geq 2$ is a harder combinatorial problem.

## 2. Notations

A tuple will always be a tuple of nonnegative integers. For such a tuple $\underline{i} = (i_0, \dots, i_s)$, let $\binom{i_0+\dots+i_s}{\underline{i}}$ be the generalized multinomial coefficient $\frac{(i_0+\dots+i_s)!}{i_0!\cdots i_s!}$.

Let $q$ be a power of a prime $p$ and $\mathbf{F}_q$ (resp. $\mathbf{F}_p$) be a finite field with $q$ (resp. $p$) elements. We will use repeatedly the following theorem of Lucas: $\binom{i_0+\dots+i_s}{\underline{i}}$ is nonzero in $\mathbf{F}_p$ if and only if there is no carry over base $p$ in the sum $i_0 + \dots + i_s$.

We keep the same notations as in the introduction. On $A = \mathbf{F}_q[T]$, we have the usual degree deg with the convention $\deg 0 = -\infty$. By convention, any ideal of $A$ that we will consider is nonzero. We will often identify an ideal $\mathfrak{p}$ of $A$ with the monic polynomial $P \in A$ generating $\mathfrak{p}$. Accordingly, $\deg \mathfrak{p}$ stands for $\deg P$.

Let $K_\infty = \mathbf{F}_q((1/T))$ be the completion of $K$ at $1/T$ with the natural nonarchimedean absolute value $|\cdot|$ such that $|T| = q$. We write $\mathbf{C}_\infty$ for the completion of an algebraic closure of $K_\infty$: it is an algebraically closed complete field for the canonical extension of $|\cdot|$ to $\mathbf{C}_\infty$.

For $P, Q$ in $A$, $(P)$ denotes the principal ideal generated by $P$, $P \mid Q$ means $P$ divides $Q$ and $(P, Q)$ is the g.c.d. of $P$ and $Q$. The integer part is denoted by $\lfloor\cdot\rfloor$.

## 3. Power sums of Carlitz coefficients

**3.1. The Carlitz module.** — Let $A\{\tau\}$ the noncommutative ring of polynomials in the indeterminate $\tau$ with coefficients in $A$ for the multiplication given by $\tau a = a^q\tau$ ($a \in A$). By the map $\tau \mapsto X^q$, the ring $A\{\tau\}$ can be identified with the subring of $\mathrm{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$ of additive polynomials of the form $\sum a_i X^{q^i}$ (where the multiplication law is given by composition). The Carlitz module is the rank-1 Drinfeld module $C : A \to A\{\tau\}$ defined by $C_T = T\tau^0 + \tau$. For $a \in A$, we put $C_a$ for the image of $a$ by $C$, as usual, and $C_a = \sum_{k=0}^{\deg a} C_{a,k}\tau^k$ with $C_{a,k} \in A$. In particular, $C_{a,0} = a$ and $C_{a,d} = 1$ if $a$ is monic of degree $d$.

**3.2. Deformation of the Carlitz module.** — We study the dependence of $C_{a,k}$ in the coefficients of $a$, when $a$ is viewed as a polynomial in $T$. For this purpose, we need a formal version of the Carlitz module. Let $\mathbf{F}_q[T, \boldsymbol{a}] = \mathbf{F}_q[T, \boldsymbol{a}_0, \boldsymbol{a}_1, \ldots]$ be the polynomial ring in $T$ and an infinite set of indeterminates $\{\boldsymbol{a}_i\}_{i\geq 0}$. Consider the ring homomorphism

$$\mathbf{C} : \mathbf{F}_q[T, \boldsymbol{a}] \longrightarrow \mathbf{F}_q[T, \boldsymbol{a}]\{\tau\}$$

defined by

$$\mathbf{C}_T = T\tau^0 + \tau, \quad \mathbf{C}_{\boldsymbol{a}_i} = \boldsymbol{a}_i\tau^0 \quad \text{for all } i \geq 0$$

where the noncommutative ring $\mathbf{F}_q[T, \boldsymbol{a}]\{\tau\}$ is defined in the obvious way. Let $P$ be an element of $\mathbf{F}_q[T, \boldsymbol{a}]$ and $d$ its degree as a polynomial in $T$. We define $\mathbf{C}_{P,0}, \ldots, \mathbf{C}_{P,d}$ in $\mathbf{F}_q[T, \boldsymbol{a}]$ by $\mathbf{C}_P = \sum_{i=0}^{d} \mathbf{C}_{P,i}\tau^i$. These coefficients satisfy the following recursive formulas.

***Lemma 3.1.*** — *Let $P \in \mathbf{F}_q[T, \boldsymbol{a}]$ be a monic of degree $d$ in $T$. Write $P = Tb + c$, with $c \in \mathbf{F}_q[\boldsymbol{a}]$ and $b \in \mathbf{F}_q[T, \boldsymbol{a}]$ monic of degree $d - 1$ in $T$. Then*

$$\begin{aligned}
\mathbf{C}_{P,0} &= T\mathbf{C}_{b,0} + c = P \\
\mathbf{C}_{P,i} &= T\mathbf{C}_{b,i} + \mathbf{C}_{b,i-1}^q \quad (1 \leq i \leq d-1) \\
\mathbf{C}_{P,d} &= \mathbf{C}_{b,d-1}^q = 1.
\end{aligned}$$

*Proof.* — Since $\mathbf{C}$ is additive, we have $\mathbf{C}_{P,i} = \mathbf{C}_{Tb,i} + \mathbf{C}_{c,i}$. Moreover, $\mathbf{C}_{c,i}$ is $c$ if $i = 0$ and $0$ otherwise. It remains to compute $\mathbf{C}_{Tb,i}$ in terms of $\mathbf{C}_{b,i}$. We have the following equalities in $\mathbf{F}_q[T, \boldsymbol{a}]\{\tau\}$:

$$\mathbf{C}_{Tb} = \mathbf{C}_T\mathbf{C}_b = (T\tau^0 + \tau)\left(\sum_{i=0}^{d-1} \mathbf{C}_{b,i}\tau^i\right) = T\left(\sum_{i=0}^{d-1} \mathbf{C}_{b,i}\tau^i\right) + \sum_{i=0}^{d-1} \mathbf{C}_{b,i}^q\tau^{i+1}.$$

By identification, we get our claim. □

***Lemma 3.2.*** — *Let $d \geq 1$ and $P \in \mathbf{F}_q[T, \boldsymbol{a}]$ monic of degree $d$ in $T$. Write $P = T^d + n_{d-1}T^{d-1} + \ldots + n_0$ with $n_0, \ldots, n_{d-1} \in \mathbf{F}_q[\boldsymbol{a}]$. For all $0 \leq j \leq d-1$, one has*

$$\mathbf{C}_{P,j} = n_j^{q^j} + TQ_j \quad \text{with } Q_j \in \mathbf{F}_q[T, n_k \mid k > j].$$

*In particular, if $P = T^d + \boldsymbol{a}_{d-1}T^{d-1} + \ldots + \boldsymbol{a}_0$, the polynomial $\mathbf{C}_{P,j}$ is independent of $\boldsymbol{a}_0$ for $j \geq 1$.*

*Proof.* — For $j=0$, we have $\mathbf{C}_{P,0} = P = n_0 + T(n_1 + \ldots + n_{d-1}T^{d-1})$ which has the expected form. For other coefficients, we proceed by induction on $d$. The statement is already proven for $d=1$. Suppose the property satisfied for all monic polynomials of degree $< d$ in $T$. Let $P = T^d + n_{d-1}T^{d-1} + \ldots + n_0$ and write $P = Tb + n_0$ with $b \in \mathbf{F}_q[T, n_1, \ldots, n_{d-1}]$ monic of degree $< d$ in $T$. Let $1 \leq j \leq d-1$. By Lemma 3.1, we have

$$\mathbf{C}_{P,j} = T\mathbf{C}_{b,j} + \mathbf{C}_{b,j-1}^q. \tag{4}$$

By hypothesis, there exists $R_{j-1} \in \mathbf{F}_q[T, n_k \mid k > j]$ and $R_j \in \mathbf{F}_q[T, n_k \mid k > j+1]$ such that $\mathbf{C}_{b,j} = n_{j+1}^{q^j} + TR_j$ and $\mathbf{C}_{b,j-1} = n_j^{q^{j-1}} + TR_{j-1}$. Substituting in (4), we get $\mathbf{C}_{P,j} = n_j^{q^j} + T(n_{j+1}^{q^j} + TR_j + T^{q-1}R_{j-1}^q)$. Since $n_{j+1}^{q^j} + TR_j + T^{q-1}R_{j-1}^q$ belongs to $\mathbf{F}_q[T, n_k \mid k > j]$, the coefficient $\mathbf{C}_{P,j}$ has the expected form. The property is then established for any monic polynomial $P$ of degree $d$. □

### 3.3. Power sums of Carlitz coefficients. —

***Notation 3.3.*** — Let $d \geq 1$. Recall that the set of monic polynomials of degree $d$ in $A$ is denoted by $A_{d+}$. For $P \in A_{d+}$ and $\underline{i} = (i_0, \ldots, i_{d-1})$, let

$$C(P)^{\underline{i}} = C_{P,0}^{i_0} \cdots C_{P,d}^{i_d} = C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}}$$

(the last equality follows from $C_{P,d} = 1$). By convention, $0^0 = 1$. Let

$$S_d(i_0, \ldots, i_{d-1}) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} \quad \in A.$$

Note that for $d=1$, the sum is just $S_1(i) = \sum_{P \in A_{1+}} P^i$. We will compute $S_d(i_0, \ldots, i_{d-1})$ for small $i_0, \ldots, i_{d-1}$.

***Lemma 3.4.*** — *Let $0 \leq i \leq 2(q-1)$ and $P \in A$. Then*

$$\sum_{a \in \mathbf{F}_q} (P+a)^i = \begin{cases} -1 & \textit{if } i = q-1 \textit{ or } 2(q-1) \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* — The vanishing case is merely an application of Lemma 3.1 of Goss [**10**]. Since we need to compute the remaining cases, we give a full proof. Let $R_i(P) = \sum_{a \in \mathbf{F}_q} (P+a)^i$. Then by the binomial formula,

$$R_i(P) = \sum_{k=0}^{i} \binom{i}{k} P^{i-k} (\sum_{a \in \mathbf{F}_q} a^k).$$

Recall that $\sum_{a \in \mathbf{F}_q} a^k$ equals $-1$ if $k > 0$ and $k \equiv 0 \mod (q-1)$, and $0$ otherwise. Thus $R_{q-1}(P) = -1$ and $R_i(P) = 0$ if $0 \leq i < q-1$. Now let $i = q+j$ with $0 \leq j \leq q-2$. Then

$$R_i(P) = \sum_{a \in \mathbf{F}_q} (P^q + a)(P+a)^j = P^q R_j(P) + \sum_{a \in \mathbf{F}_q} a(P+a)^j.$$

Since $j \leq q-2$, $R_j(P)$ is zero. Moreover, by the binomial formula,

$$\sum_{a\in\mathbf{F}_q} a(P+a)^j = \sum_{k=0}^{j} \binom{j}{k} P^{j-k}(\sum_{a\in\mathbf{F}_q} a^{k+1})$$

which is 0 if $j < q-2$ (resp. $-1$ if $j = q-2$). □

***Proposition 3.5***. — *Let $i_j \in \{0,\ldots,2(q-1)\}$ for all $j \in \{0,\ldots,d-1\}$. Then*

$$S_d(i_0,\ldots,i_{d-1}) = \begin{cases} (-1)^d & \textit{if, for all } j,\ i_j \in \{q-1, 2(q-1)\} \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof*. — The sum $S_d(i_0,\ldots,i_{d-1})$ is equal to

$$\sum_{a_0,\ldots,a_{d-1}\in\mathbf{F}_q} C^{i_0}_{T^d+a_{d-1}T^{d-1}+\cdots+a_0,0}\cdots C^{i_{d-1}}_{T^d+a_{d-1}T^{d-1}+\cdots+a_0,d-1}.$$

By Lemma 3.2, the polynomials $C_{T^d+\cdots+a_0,1},\ldots,C_{T^d+\cdots+a_0,d-1}$ do not depend on $a_0$, so we can rewrite the sum as

$$\sum_{a_1,\ldots,a_{d-1}\in\mathbf{F}_q} C^{i_1}_{T^d+\cdots+a_1T,1}\cdots C^{i_{d-1}}_{T^d+\cdots+a_1T,d-1}\left(\sum_{a_0\in\mathbf{F}_q}(T^d+\ldots+a_1T+a_0)^{i_0}\right).$$

Let $\epsilon_j$ be $-1$ if $i_j \in \{q-1, 2(q-1)\}$ and 0 otherwise. Since $0 \leq i_0 \leq 2(q-1)$, Lemma 3.4 gives $\sum_{a_0\in\mathbf{F}_q}(T^d+\ldots+a_1T+a_0)^{i_0} = \epsilon_0$. Then, again by Lemma 3.2, $S_d(i_0,\ldots,i_{d-1})$ is equal to

$$\epsilon_0 \sum_{a_2,\ldots,a_{d-1}\in\mathbf{F}_q} C^{i_2}_{T^d+\cdots+a_2T^2,2}\cdots C^{i_{d-1}}_{T^d+\cdots+a_2T^2,d-1}\left(\sum_{a_1\in\mathbf{F}_q}(TQ_1+a_1^q)^{i_1}\right).$$

Since $0 \leq i_1 \leq 2(q-1)$, Lemma 3.4 yields $\sum_{a_1\in\mathbf{F}_q}(TQ_1+a_1^q)^{i_1} = \sum_{a_1\in\mathbf{F}_q}(TQ_1+a_1)^{i_1} = \epsilon_1$. Continuing in this fashion, we obtain $S_d(i_0,,\ldots,i_{d-1}) = \epsilon_0\cdots\epsilon_{d-1}$. □

**3.4. Connection with Carlitz binomial coefficients and special zeta values.** — We recall Carlitz's analogue $\left\{ {a \atop k} \right\}$ in $\mathbf{F}_q[T]$ of the binomial coefficient $\binom{n}{k}$ (the reader may consult Thakur's article [**21**] for examples of such analogies). Let $a \in A$ and $k \in \mathbf{N}$ with base $q$ expansion $\sum_{i=0}^{w} k_i q^i$ $(0 \leq k_i < q)$. We put $\left\{ {a \atop k} \right\} = \prod_{i=0}^{w} C_{a,i}^{k_i}$ (if $i > \deg a$, $C_{a,i} = 0$ by convention). In particular, $\left\{ {a \atop q^i} \right\} = C_{a,i}$. Note that if $0 \leq i_j < q$, then

$$C(P)^{\underline{i}} = C^{i_0}_{P,0}\ldots C^{i_{d-1}}_{P,d-1} = \left\{ {P \atop i_0+i_1+\ldots+i_{d-1}q^{d-1}} \right\}.$$

In general ($i_j \geq q$), it is still possible to write $C^{i_0}_{P,0}\ldots C^{i_{d-1}}_{P,d-1}$ in terms of several Carlitz binomials. We now explain how Proposition 3.5 might be proved using this formalism.

If $x$ is an indeterminate, $\left\{ {x \atop k} \right\}$ is a polynomial in $K[x]$ with degree $k$ (because $\left\{ {x \atop k} \right\}$ is also the exponential function of a finite lattice, see Equation 2.5 of [**21**] or [**14**]). Any polynomial $f$ in $K[x]$ may therefore be written as a linear combination of $\left\{ {x \atop k} \right\}$. Moreover, the coefficients of this combination can be recovered, in terms of $\left\{ {x \atop k} \right\}$, by a Mahler inversion type formula due to Carlitz (Theorem 6 in [**4**], Lemma 3.2.14 in [**14**] or Theorem XIV in [**21**]). For $f = 1$, the coefficients in the binomial basis are easily computable and, by the inversion, we obtain for $d \geq 0$ and $0 \leq i < q^d$ with base $q$ expansion $\sum_{j=0}^{d-1} i_j q^j$,

$$S_d(i_0, \dots, i_{d-1}) = \sum_{P \in A_{d+}} \left\{ \begin{matrix} P \\ i \end{matrix} \right\} = \begin{cases} (-1)^d & \text{if } i = q^d - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is precisely a special case of Proposition 3.5 (see also [**21**] p. 14 and Theorem 3.2.16 in [**14**] for similar statements). It seems likely that Proposition 3.5 can be proved by Mahler inversion.

Finally, we explain how, by the previous observations, $S_d(i_0, \dots, i_{d-1})$ is related to special zeta values of Goss zeta function at negative integers. Consider the Carlitz zeta function $\zeta : \mathbf{N} \to K_\infty$ defined by $\zeta(k) = \sum_{P \in A, P \text{ monic}} P^{-k}$. In [**10**] Goss proved that $\zeta$ can be extended to $\mathbf{Z}$ by summing over fixed degree: $\zeta(-k) = \sum_{i=0}^{\infty} (\sum_{P \in A_{i+}} P^k) \in A$ for $k \geq 0$. Now, let $\mathfrak{p}$ be a prime ideal of $A$ and $A_\mathfrak{p}$ the ring of integers of the completion of $K$ at $\mathfrak{p}$. Following Thakur [**21**], one can attach to $\zeta$ an $A_\mathfrak{p}$-valued zeta measure $\mu$ determined by its $k$th moment:

$$\int_{A_\mathfrak{p}} x^k d\mu = \begin{cases} \zeta(-k) & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases}$$

By Wagner's Mahler-inversion formula for continous functions on $A_\mathfrak{p}$ ([**14**] or Theorem VI in [**21**]), the measure $\mu$ is uniquely determined by the coefficients of its divided power series i.e. the sequence $\mu_k = \int_{A_\mathfrak{p}} \{ \begin{smallmatrix} x \\ k \end{smallmatrix} \} d\mu$ ($k \geq 0$). Thakur has computed explicitly $\mu_k$ ([**21**], Theorem VII). It follows from his proof that, when $0 \leq i_j < q$ and $i = i_0 + \ldots + i_{d-1} q^{d-1}$,

$$S_d(i_0, \dots, i_{d-1}) = \mu_{i+q^d}.$$

## 4. Drinfeld modular forms and Hecke operators

We collect some basic facts, and set up notation and terminology as well, for Drinfeld modular forms and Hecke operators.

**4.1. Drinfeld modular forms.** — The first occurrence of Drinfeld modular forms goes back to the seminal work of D. Goss [**12, 13**]. Subsequent developments in the 1980s are due to Gekeler [**5, 7**].

The so-called Drinfeld upper-half plane is $\Omega = \mathbf{C}_\infty - K_\infty$, which has a rigid analytic structure. For an ideal $\mathfrak{n}$ of $A$, the Hecke congruence subgroup $\Gamma_0(\mathfrak{n})$ is the subgroup of matrices $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(A)$ such that $c \in \mathfrak{n}$. Fix an integer $k \geq 0$ and a class $m$ in $\mathbf{Z}/(q-1)\mathbf{Z}$. From now on, $m$ will denote the unique representative of such a class in $\{0, 1, \dots, q-2\}$. A *Drinfeld modular form* (for $\Gamma_0(\mathfrak{n})$) of weight $k$ and type $m$ is a rigid holomorphic function $f : \Omega \to \mathbf{C}_\infty$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (ad-bc)^{-m}(cz+d)^k f(z) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{n}) \tag{5}$$

and $f$ is holomorphic at the cusps of $\Gamma_0(\mathfrak{n})$. We will not detail the second assumption and rather refer to [**5**] (V, Section 3) and [**15**] (Section 2). For our purpose, we need only the behaviour at the cusp infinity, which we now recall.

Let $\overline{\pi}$ be the period of the Carlitz module (well-defined up to multiplication by an element in $\mathbf{F}_q^\times$). The Carlitz exponential $e$ is the holomorphic function $\mathbf{C}_\infty \to \mathbf{C}_\infty$ defined by

$$e(z) = z \prod_{\lambda \in \overline{\pi}A - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

It is surjective and $\mathbf{F}_q$-linear with kernel $\overline{\pi}A$. For $z \in \mathbf{C}_\infty - A$, let

$$t(z) = \frac{1}{e(\overline{\pi}z)} = \frac{1}{\overline{\pi}} \sum_{\lambda \in A} \frac{1}{z - \lambda}.$$

The function $t$, invariant by translations $z \mapsto z + a$ ($a \in A$), is then a uniformizer at the cusp infinity. Since any $f$ satisfying (5) is invariant under such translations, it has a Laurent series expansion $f(z) = \sum_{i \geq i_0} a_i(f) t(z)^i$ with $i_0 \in \mathbf{Z}$ (the series does not converge on all $\Omega$, but only for $|t(z)|$ small enough). Such a function is said to be *holomorphic at the cusp infinity* if the expansion has the form $\sum_{i \geq 0} a_i(f) t^i$. We call it the *$t$-expansion of $f$* (at infinity). Since $\Omega$ is a connected rigid analytic space, any Drinfeld modular form is uniquely determined by its $t$-expansion.

Let $M_{k,m}(\Gamma_0(\mathfrak{n}))$ be the space of Drinfeld modular forms of weight $k$ and type $m$ for $\Gamma_0(\mathfrak{n})$. It is a finite-dimensional vector space over $\mathbf{C}_\infty$ whose dimension may be calculated explicitly thanks to Gekeler [**5**]. If $a_0(f) = 0$ (resp. $a_0(f) = a_1(f) = 0$) and similar conditions at other cusps, $f$ is *cuspidal* (resp. *doubly cuspidal*) and the subspace of such functions is denoted by $M^1_{k,m}(\Gamma_0(\mathfrak{n}))$ (resp. $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$). Goss observed that doubly cuspidal forms play a role similar to classical cusp forms. For an interpretation of Drinfeld modular forms as differentials on a Drinfeld modular curve, one may refer to Section V.5 in [**5**].

Type and weight are not independent: namely, if $k - 2m \not\equiv 0 \mod (q-1)$, the space $M_{k,m}(\Gamma_0(\mathfrak{n}))$ is zero. Therefore, from now on we assume $k \equiv 2m \mod (q-1)$.

Since $\Gamma_0(\mathfrak{n})$ contains matrices of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $\lambda \in \mathbf{F}_q^\times$, (5) implies $a_i(f) = 0$ when $i \not\equiv m \mod (q-1)$. Thus any $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ has $t$-expansion of the form

$$\sum_{j \geq 0} a_{m+j(q-1)}(f) t^{m+j(q-1)}.$$

For $j \geq 0$, let

$$b_j(f) = a_{m+j(q-1)}(f).$$

Later on, we will use both notations for coefficients. A Drinfeld modular form of type $> 0$ (resp. $> 1$) is automatically cuspidal (resp. doubly cuspidal). If $f$ is doubly cuspidal, the coefficient $b_0(f)$ may not vanish in general (it does when $m \in \{0, 1\}$).

**4.2. Hecke algebra.** — We define a formal Hecke algebra $\mathbf{R}_\mathfrak{n}$ which acts on the different spaces $M_{k,m}(\Gamma_0(\mathfrak{n}))$. In this section, we adopt the notation $\Gamma = \Gamma_0(\mathfrak{n})$.

Let $\Delta = \Delta_0(\mathfrak{n})$ be the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in $A$ such that $ad - bc$ is monic, $c \in \mathfrak{n}$ and $(a) + \mathfrak{n} = A$. Let $\mathbf{R}_\mathfrak{n}$ be the $\mathbf{C}_\infty$-vector space spanned by formal sums of double cosets $\Gamma g \Gamma$ for $g \in \Delta$. One can make $\mathbf{R}_\mathfrak{n}$ a commutative algebra over $\mathbf{C}_\infty$ (see Section 3.1 of [**17**] for a general treatment or Section 6.1 of [**3**] for Drinfeld modular forms).

For an ideal $\mathfrak{p}$ of $A$, let $\Delta^{\mathfrak{p}}$ be the set of $g \in \Delta$ such that $\det g$ is the monic generator of $\mathfrak{p}$. The Hecke operator $T_{\mathfrak{p}}$ is then defined as the formal sum of all double cosets $\Gamma g \Gamma$ with $g \in \Delta^{\mathfrak{p}}$ in $\mathbf{R}_{\mathfrak{n}}$. For example, when $\mathfrak{p}$ is prime, $T_{\mathfrak{p}} = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & P \end{smallmatrix}\right)\Gamma$ where $P$ is the monic generator of $\mathfrak{p}$.

As elements of $\mathbf{R}_{\mathfrak{n}}$ have coefficients in a field of characteristic $p$, the usual relation for the product gives

$$T_{\mathfrak{p}} T_{\mathfrak{p}'} = T_{\mathfrak{p}\mathfrak{p}'} \quad \text{for } \textit{any} \text{ ideals } \mathfrak{p}, \mathfrak{p}'$$

(see [**11**]). This is very different from Hecke operators for classical modular forms, where the above relation only holds for relatively prime ideals. One can check that $\mathbf{R}_{\mathfrak{n}}$ is the polynomial ring over $\mathbf{C}_\infty$ spanned by $T_{\mathfrak{p}}$ for $\mathfrak{p}$ prime (such elements are algebraically independent over $\mathbf{C}_\infty$).

As for the notation, $T_{\mathfrak{p}}$ depends on the subgroup $\Gamma_0(\mathfrak{n})$ but from the context, there will be no confusion on which Hecke algebra (or space of endomorphisms of Drinfeld modular forms) it belongs to.

For $\mathfrak{n} = A$, let us consider the formal Hecke algebra $\mathbf{R}_A$ attached to the sets $\mathrm{GL}_2(A)$ and $\Delta_0(A)$. Let $\tilde{T}_{\mathfrak{p}}$ temporarily denotes the $\mathfrak{p}$th Hecke operator in $\mathbf{R}_A$. The map $\tilde{T}_{\mathfrak{p}} \mapsto T_{\mathfrak{p}}$, for $\mathfrak{p}$ prime, defines an algebra homomorphism $\mathbf{R}_A \to \mathbf{R}_{\mathfrak{n}}$. We regard $\mathbf{R}_A$ as a universal formal Hecke algebra, independent of $\mathfrak{n}$. Any algebraic relation among the Hecke operators in $\mathbf{R}_A$ can be translated to the corresponding relation in $\mathbf{R}_{\mathfrak{n}}$ for any $\mathfrak{n}$.

**4.3. Hecke operators on Drinfeld modular forms.** — For $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in $A$ and $f : \Omega \to \mathbf{C}_\infty$, let

$$f_{|[v]_k} : z \longmapsto (ad-bc)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Fix $g \in \Delta$. The group $\Gamma$ acts on the left on the double coset $\Gamma g \Gamma$. Let $\{g_i\}_i$ be a finite system of representatives such that $g_i$ has monic determinant. We define an action of $\Gamma g \Gamma$ on $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ by

$$f_{|[\Gamma g \Gamma]_k} = \sum_i f_{|[g_i]_k}$$

(independently of the choice of $\{g_i\}_i$). It extends, in a unique way, to a non-faithful action of $\mathbf{R}_{\mathfrak{n}}$ on $M_{k,m}(\Gamma_0(\mathfrak{n}))$. Let $\mathbf{T} = \mathbf{T}_{k,m}(\Gamma_0(\mathfrak{n}))$ be the commutative sub-$\mathbf{C}_\infty$-algebra of $\mathrm{End}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})))$ induced by $\mathbf{R}_{\mathfrak{n}}$.

For any $g \in \Delta^{\mathfrak{p}}$, a set of representatives of $\Gamma\backslash\Gamma g \Gamma$ with monic determinant is given by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \quad \alpha, \delta \text{ monic in } A, (\alpha\delta) = \mathfrak{p}, (\alpha) + A = \mathfrak{n}, \beta \in A/(\delta).$$

Therefore, the action of $T_{\mathfrak{p}}$ on the Drinfeld modular form $f$ can be written more concretely as

$$(6) \qquad T_{\mathfrak{p}}(f)(z) = P^{k-1} \sum_{\substack{\alpha,\delta \text{ monic } \in A \\ \beta \in A,\ \deg\beta < \deg\delta \\ \alpha\delta = P, (\alpha)+\mathfrak{n} = A}} \delta^{-k} f\left(\frac{\alpha z+\beta}{\delta}\right) = \frac{1}{P} \sum_{\alpha,\beta,\delta} \alpha^k f\left(\frac{\alpha z+\beta}{\delta}\right)$$

where $P$ is the monic generator of $\mathfrak{p}$. This formula slightly differs from other references. Gekeler [**7**] (resp. Böckle [**3**], Section 6) considered $PT_{\mathfrak{p}}$ (resp. $P^{m+1-k}T_{\mathfrak{p}}$). In particular, our operator coincides with Böckle's when $k = m-1$ (for instance, when $k = 2$ and

$m = 1$). In general, these variously normalized Hecke operators have the same eigenforms, however with different eigenvalues.

The Hecke algebra $\mathbf{T}$ stabilizes the subspaces $M^1_{k,m}(\Gamma_0(\mathfrak{n}))$ et $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ (see for example Proposition 6.9 of [**3**]). We denote by $\mathbf{T}' = \mathbf{T}'_{k,m}(\Gamma_0(\mathfrak{n}))$ the restriction of $\mathbf{T}$ to $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$.

## 5. Hecke action on the first coefficient of Drinfeld modular forms

We recall some results on Goss polynomials for finite lattices and their role in the $t$-expansion of Drinfeld modular forms. Then we give an explicit formula for the action of Hecke operators on the first coefficient of this expansion.

**5.1. Goss polynomials.** — Let $\Lambda$ be a $\mathbf{F}_q$-lattice in $\mathbf{C}_\infty$, i.e. a $\mathbf{F}_q$-submodule of $\mathbf{C}_\infty$ having finite intersection with each ball of $\mathbf{C}_\infty$ of finite radius. We assume $\Lambda$ to be *finite* of dimension $d$ over $\mathbf{F}_q$. The exponential corresponding to $\Lambda$

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right) \qquad (z \in \mathbf{C}_\infty)$$

is an entire $\Lambda$-periodic $\mathbf{F}_q$-linear function. Since $\Lambda$ is finite, it is a polynomial in $z$ of the form

$$e_\Lambda(z) = \sum_{i=0}^{d} \lambda_i z^{q^i}$$

with coefficients $\lambda_i \in \mathbf{C}_\infty$ depending on $\Lambda$. Goss has considered Newton's sums associated to the reciprocal polynomial of $e_\Lambda(X - z) = e_\Lambda(X) - e_\Lambda(z) \in \mathbf{C}_\infty[z][X]$, namely

$$\begin{aligned}
&N_0 = 0\\
&N_j(z) = N_{j,\Lambda}(z) = \textstyle\sum_{\lambda \in \Lambda} \frac{1}{(z+\lambda)^j} \qquad (j \geq 1, z \in \mathbf{C}_\infty - \Lambda).
\end{aligned}$$

Let

$$t_\Lambda(z) = e_\Lambda(z)^{-1} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} \qquad (z \in \mathbf{C}_\infty - \Lambda).$$

***Proposition 5.1*** **(Section 2 of [13], 3.4–3.9 in [7]).** — *Let $j \geq 1$. There exists a unique polynomial $G_j = G_{j,\Lambda}(X) \in \mathbf{C}_\infty[X]$ such that the following equalities hold:*

1. *if $j \leq q$ then $G_j(X) = X^j$*
2. *$G_j(X) = X \sum_{i \geq 0, j - q^i \geq 0} \lambda_i G_{j-q^i}(X)$.*

*The polynomial $G_j(X)$ is monic of degree $j$ and satisfies $N_j = G_j(t_\Lambda)$. Moreover,*

$$G_j(X) = \sum_{n=0}^{j-1} \sum_{\underline{i}} \tbinom{n}{\underline{i}} \lambda^{\underline{i}} X^{n+1} \tag{7}$$

*for $\underline{i} = (i_0, \ldots, i_d)$ running through $(d+1)$-tuples such that*

$$\begin{aligned}
i_0 + \ldots + i_d &= n\\
i_0 + i_1 q + \ldots + i_d q^d &= j - 1
\end{aligned}$$

*and $\lambda^{\underline{i}}$ denotes $\lambda_0^{i_0}\cdots\lambda_d^{i_d}$. The polynomial $G_j(X)$ is divisible by $X^u$ where $u=\lfloor j/q^d\rfloor+1$. We further put $G_{0,\Lambda}(X)=0$.*

Gekeler provided the explicit formula (7) using a generating function.

**5.2. Hecke algebra and Goss polynomials.** — Let $\mathfrak{p}$ an ideal of $A$ of degree $d\geq 0$ with monic generator $P$. Recall that $C$ denotes the Carlitz module over $\mathbf{C}_\infty$ (Section 3.1). As usual, for an indeterminate $X$, put $C_P(X)=\sum_{i=0}^d C_{P,i}X^{q^i}$. For our purpose, we consider the $\mathbf{F}_q$-lattice of dimension $d$

$$\Lambda_P=\mathrm{Ker}(C_P)=\{x\in\mathbf{C}_\infty \mid C_P(x)=0\}$$

whose $j$th Goss polynomial is denoted by $G_{j,P}$. Let

$$t_P(z)=t(Pz)=\frac{1}{e(\overline{\pi}Pz)}\qquad (z\in\mathbf{C}_\infty-A).$$

Then, if $f_P(X)$ is the $P$th inverse cyclotomic polynomial $C_P(X^{-1})X^{q^d}$, one has

$$t_P=\frac{t^{q^d}}{f_P(t)}. \tag{8}$$

The following statement mildly extends Gekeler's formula 7.3 in [**7**] (which was established for $\mathrm{GL}_2(A)$ and $\mathfrak{p}$ prime) to $\Gamma_0(\mathfrak{n})$ and any $\mathfrak{p}$.

***Proposition 5.2.*** — *Let $f\in M_{k,m}(\Gamma_0(\mathfrak{n}))$ with $t$-expansion $\sum_{i\geq 0}a_it^i$. We have*

$$T_{\mathfrak{p}}f=P^{k-1}\sum_{i\geq 0}\ \sum_{\substack{\delta \text{ monic in } A\\ \delta\mid P,\ (\frac{P}{\delta})+\mathfrak{n}=A}}\delta^{-k}a_iG_{i,\delta}(\delta t_{\frac{P}{\delta}}) \tag{9}$$

*Moreover, for fixed $j$, only a finite number of terms in the right-hand side contribute to $t^j$ in the $t$-expansion of $T_{\mathfrak{p}}f$.*

*Proof.* — Let $\delta$ be a monic divisor of $P$. Recall that $e$ is the Carlitz exponential. We write $F(z)$ for $\sum_{\beta\in A,\deg\beta<\deg\delta}f\left((Pz/\delta+\beta)/\delta\right)$. For $|t(z)|$ small enough, $F(z)$ is

$$\begin{aligned}\sum_{\beta\in A,\deg\beta<\deg\delta}\ \sum_{i\geq 0}a_it\left(\frac{\frac{P}{\delta}z+\beta}{\delta}\right)^i&=\sum_{i\geq 0}a_i\sum_{\beta\in A,\deg\beta<\deg\delta}e\left(\overline{\pi}\frac{\frac{P}{\delta}z+\beta}{\delta}\right)^{-i}\\&=\sum_{i\geq 0}a_i\sum_{\beta\in A,\deg\beta<\deg\delta}\left(e\left(\frac{\overline{\pi}Pz}{\delta^2}\right)+e\left(\frac{\overline{\pi}\beta}{\delta}\right)\right)^{-i}\end{aligned}$$

by additivity of $e$. According to the analytic theory of Drinfeld modules, the finite set $\{e(\overline{\pi}\beta/\delta)\mid \beta\in A,\deg\beta<\deg\delta\}$ is in bijection with the lattice $\Lambda_\delta=\mathrm{Ker}(C_\delta)$. Let $w=Pz/\delta^2$. Then, by Proposition 5.1, $F(z)$ is

$$\sum_{i\geq 0}a_i\sum_{\lambda\in\Lambda_\delta}(e(\overline{\pi}w)+\lambda)^{-i}=\sum_{i\geq 0}a_iN_{i,\Lambda_\delta}(e(\overline{\pi}w))=\sum_{i\geq 0}a_iG_{i,\Lambda_\delta}(e_{\Lambda_\delta}(e(\overline{\pi}w))^{-1}).$$

Observe that $e_{\Lambda_\delta}(z)=C_\delta(z)/\delta$ (both polynomials have the same set of zeros and the multiplicative constant is obtained by comparing the terms in $z$). By the properties of the Carlitz exponential, we also have $C_\delta(e(\overline{\pi}w)) = C(\overline{\pi}zP/\delta) = t(zP/\delta)^{-1}$. Substituting, we get

$$F(z) = \sum_{i\geq 0} a_i G_{i,\Lambda_\delta}\left(\delta t\left(\frac{zP}{\delta}\right)\right) = \sum_{i\geq 0} a_i G_{i,\Lambda_\delta}(\delta t_{\frac{P}{\delta}}(z)).$$

Our claim follows from (6) and the last statement of Proposition 5.1. □

***Remark 5.3***. — To obtain the $t$-expansion of $T_{\mathfrak{p}}f$ from Equation (9), it would suffice to compose the $t$-expansions of $t_{P/\delta}$ and Goss polynomials $G_{i,\delta}$. However, making this explicit seems to be a difficult problem. Indeed, a similar question arises when trying to make explicit the $t$-expansion of Drinfeld-Eisenstein series (see Section 6 of [**7**]) since it involves the $t$-expansion of $G_{i,\overline{\pi}A}(t_P)$(1). This is quite different from the classical situation where coefficients of Eisenstein series are well-known arithmetic functions.

### 5.3. Hecke module spanned by $b_1$. —

***Notation 5.4***. — The dual space of $M_{k,m}(\Gamma_0(\mathfrak{n}))$ has the natural right action of $\mathbf{T}$, given by composition, and contains the following linear forms, for any $n\geq 1$:

$$a_{m+n(q-1)} = b_n : f \mapsto a_{m+n(q-1)}(f) = b_n(f).$$

Let $u = u_{k,m,\mathfrak{n}} : \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ be the $\mathbf{C}_\infty$-linear map $s \mapsto b_1 s$. We write $b_1\mathbf{T}'$ for the image of $u$.

We collect some remarks on the dimension of the $\mathbf{C}_\infty$-algebra $\mathbf{T}'$. The map $u$ is not necessarily an isomorphism, therefore the dimension of $\mathbf{T}'$ is unknown *a priori.* In the case $\mathbf{T}' = \mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$, one can prove that its dimension coincides with $\dim_{\mathbf{C}_\infty} M^2_{2,1}(\Gamma_0(\mathfrak{n}))$, using results from automorphic forms and work of Gekeler and Reversat [**15**].

We keep Notation 3.3. The next statement gives a first description of $b_1\mathbf{T}'$.

***Proposition 5.5***. — *Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ with $t$-expansion $\sum_{i\geq 0} a_i(f)t^i$. Let $\mathfrak{p}$ an ideal of $A$ of degree $d$ with monic generator $P$. Then $a_{m+(q-1)}(T_{\mathfrak{p}}f)$ is*

$$\sum_{\underline{n}} \binom{m+q-2}{\underline{n}} C(P)^{\underline{n}} a_{1+n_0+n_1q+\ldots+n_dq^d}(f) + \varepsilon \sum_{\substack{Q|P, Q\in A_{1+} \\ (Q)+\mathfrak{n}=A}} Q^{k-1}a_1(f) \tag{10}$$

*where $\underline{n} = (n_0,\ldots,n_d)$ is such that $n_0+\ldots+n_d = m+q-2$ with $n_i\geq 0$ for all $i$ and $\varepsilon$ is defined by* $\varepsilon = \begin{cases} 1 & \text{if } m=1 \\ 0 & \text{otherwise.}\end{cases}$

***Remark 5.6***. —
1. In Example 7.4 of [**7**], Gekeler treated $a_i(T_{\mathfrak{p}}f)$ for $\mathfrak{p}$ of degree 1, $i\geq 0$, and $f$ modular for $\mathrm{GL}_2(A)$. Proposition 5.5 supplements Gekeler's statement.
2. Actually, Propositions 5.2 and 5.5 work for holomorphic functions $f : \Omega \to \mathbf{C}_\infty$ having an expansion $\sum_{i\geq 0} a_i t^i$ for $|t(z)|$ small enough (Hecke operators are still defined on $f$ via (6)). In particular, this applies to holomorphic functions $f : \Omega \to \mathbf{C}_\infty$ which are $A$-periodic ($f(z+a) = f(z)$, $a\in A$) and holomorphic at the cusp infinity.

(1)The lattice $\overline{\pi}A$ is not finite but Goss polynomials can be defined in that more general setting (see [**13, 7**]).

*Proof.* — By Proposition 5.2, we have to find the coefficient of $t^{m+(q-1)}$ in the $t$-expansion of $G_{i,\delta}(\delta t_{P/\delta})$. First, if $i=0$, then $G_{0,\delta}(X)=0$ so the expansion of $G_{0,\delta}(\delta t_{P/\delta})$ has no $t^{m+(q-1)}$-term.

Assume $i>0$. By (8) the $t$-expansion of $t_{P/\delta}$ is divisible by $t^{q^{d-\deg\delta}}$. Moreover, it follows from the definition of Goss polynomials that $G_{i,\delta}(X)$ has $X$ as a factor. Hence, the $t$-expansion of $G_{i,\delta}(\delta t_{P/\delta})$ is divisible by $t^{q^{d-\deg\delta}}$. Since $m<q-1$, $G_{i,\delta}(\delta t_{P/\delta})$ has no $t^{m+(q-1)}$-term when $d-\deg\delta\geq 2$. Now assume $0\leq d-\deg\delta\leq 1$. Put $s=\deg\delta$. Recall that $e_{\Lambda_\delta}(z)=C_\delta(z)/\delta=\sum_{i=0}^{s}C_{\delta,i}z^{q^i}/\delta$. The explicit formula for Goss polynomials gives

$$G_{i,\delta}(X)=\sum_{j=0}^{i-1}\delta^{-j}\sum_{\underline{n}}\tbinom{j}{\underline{n}}C(\delta)^{\underline{n}}X^{j+1}$$

where $\underline{n}=(n_0,\ldots,n_s)$ are such that $n_0+\ldots+n_s=j$ and $n_0+n_1q+\ldots+n_sq^s=i-1$.

Suppose that $s=d$, i.e. $\delta=P$. Then the corresponding partial sum in (9) is

$$\frac{1}{P}\sum_{i\geq 0}a_iG_{i,P}(Pt)=\frac{1}{P}\sum_{i\geq 0}a_i\sum_{j=0}^{i-1}P^{-j}\sum_{\underline{n}}\tbinom{j}{\underline{n}}C(P)^{\underline{n}}(Pt)^{j+1}.$$

The $t^{m+(q-1)}$-term corresponds to $j=m+q-2$; namely, it is

$$\sum_{\underline{n}}\tbinom{m+q-2}{\underline{n}}C(P)^{\underline{n}}a_{1+n_0+n_1q+\ldots+n_dq^d}(f)$$

with $\underline{n}=(n_0,\ldots,n_d)$ such that $n_0+\ldots+n_d=m+q-2$.

Next, suppose that $s=d-1$. Using Equation (8), we get

$$G_{i,\delta}(\delta t_{\frac{P}{\delta}})=\sum_{j=0}^{i-1}\delta^{-j}\sum_{\underline{n}}\tbinom{j}{\underline{n}}C(\delta)^{\underline{n}}\left(\delta\frac{t^q}{1+\frac{P}{\delta}t^{q-1}}\right)^{j+1}\tag{11}$$

where $(n_0,\ldots,n_{d-1})$ with $n_0+\ldots+n_{d-1}=j$ and $n_0+n_1q+\ldots+n_{d-1}q^{d-1}=i-1$. If $j\geq 1$, then $q(j+1)\geq 2q>m+q-1$, thus there is no $t^{m+(q-1)}$-term in the expansion of (11). Finally, we assume $j=0$, in other words $n_0=\ldots=n_{d-1}=0$ and $i=1$. We have

$$G_{1,\delta}(\delta t_{\frac{P}{\delta}})=\delta\frac{t^q}{1+\frac{P}{\delta}t^{q-1}}=\delta t^q\sum_{n=0}^{+\infty}(-1)^n\frac{P^n}{\delta^n}t^{n(q-1)}.$$

This series has a $t^{m+(q-1)}$-term if and only if $q-1$ divides $m-1$. This happens only if $m=1$, and in that case the corresponding coefficient is $\delta$. Thus we obtain (10). □

Assume $m\in\{0,1\}$. By (10), the linear form $b_1T_{\mathfrak{p}}=a_{m+(q-1)}T_{\mathfrak{p}}$ is a $A$-linear combination of $a_i$, where $i$ runs through the set of natural integers satisfying the condition: the expansion of $i$ in base $q$ has at most $d+1$ digits, whose sum is equal to $m+q-1$. In particular, the set of such $i$'s only depends on the degree $d$ of $\mathfrak{p}$. This observation, also communicated to the author by D. Goss, will be used in Section 7. For the moment, we derive the following statement for $b_1\mathbf{T}'$.

***Notation 5.7.*** — Let $\mathscr{S}$ be the set of natural integers of the form $c/(q-1)$ where $c\in\mathbf{N}$ is such that the sum of its base $q$ digits is $q-1$.

***Corollary 5.8.*** — *If $m \in \{0,1\}$ then $b_1\mathbf{T}'$ is contained in the $\mathbf{C}_\infty$-vector space spanned by $b_n$ for $n \in \mathscr{S}$.*

The reverse inclusion will be proved in Section 7. Finally, we state another straightforward application of Proposition 5.5.

***Notation 5.9.*** — For $d \geq 1$ and $\underline{i} = (i_0, \ldots, i_{d-1})$, let

$$\Theta_d(i_0, \ldots, i_{d-1}) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} T_P = \sum_{P \in A_{d+}} C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}} T_P \quad \in \mathbf{R}_A.$$

***Corollary 5.10.*** — *Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$. With the notations of Proposition 5.5 and Section 3, the coefficient $a_{m+(q-1)}(\Theta_d(i_0, \ldots, i_{d-1})f)$ is*

$$\sum_{\substack{\underline{n}=(n_0,\ldots,n_d) \\ n_0+\ldots+n_d=m+q-2}} \binom{m+q-2}{\underline{n}} S_d(n_0+i_0, \ldots, n_{d-1}+i_{d-1}) a_{1+n_0+n_1q+\ldots+n_dq^d}(f)$$
$$+\varepsilon \sum_{P \in A_{d+}} C(P)^{\underline{i}} \sum_{\substack{Q|P, Q \in A_{1+} \\ (Q)+\mathfrak{n}=A}} Q^{k-1} a_1(f)$$

*where $\varepsilon$ is defined as in Proposition 5.5.*

## 6. Annihilator of $b_1$ for the Hecke action

***Notation 6.1.*** — Let $\mathbf{I} = \mathbf{I}_{k,m,\mathfrak{n}}$ be the kernel of $u$ i.e. the ideal of elements $s \in \mathbf{T}'$ such that $b_1 s = 0$ in the dual space of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$.

In particular, $\mathbf{I}$ is a sub-$\mathbf{C}_\infty$-algebra of $\mathbf{T}'$ which maps doubly cuspidal forms to Drinfeld modular forms $f$ satisfying $a_0(f) = b_0(f) = b_1(f) = 0$.

***Lemma 6.2.*** — *If the map $u : \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ is surjective, then it is an isomorphism.*

*Proof.* — Since $u$ is surjective, we take an element $t_n$ in the preimage of $b_n$ for any $n \geq 1$. If $s$ belongs to the ideal $\mathbf{I}$, so does $t_n s$. Hence, for any $f \in M^2_{k,m}(\Gamma_0(\mathfrak{n}))$, the $n$th coefficient $b_n(sf)$ is zero for any $n \geq 1$. As the $t$-expansion uniquely determines a Drinfeld modular form, $sf$ must be zero. Therefore $s$ is zero as an endomorphism of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$. □

### 6.1. Proof of Theorem 1.1. —

*Proof of Theorem 1.1.* — Actually we prove a slightly more general statement: all the following equalities of linear forms will take place in the dual space of $M_{k,m}(\Gamma_0(\mathfrak{n}))$ if $m \neq 1$ (resp. of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ if $m = 1$).

1. Without any assumption on $m$, we apply Corollary 5.10 to $d = 1$. For $i \geq 0$ we get

$$b_1\left(\sum_{P \in A_{1+}} P^i T_P\right) = \sum_{n=0}^{m+q-2} \binom{m+q-2}{n} S_1(n+i) a_{1+n+q(m+q-2-n)}.$$

This follows also from Gekeler's example 7.4 in [**7**], although stated there for $\mathrm{GL}_2(A)$ and with a different normalization of Hecke operators.

Assume $m=0$. The sum $S_1(n+1)=\sum_{Q\in A_{1+}} Q^{n+1}$ is nonzero if and only if $n=q-2$, and $S_1(q-1)=-1$ (by Lemma 3.4 for instance). Taking $i=1$, our expression simplifies as $b_1\left(\sum_{P\in A_{1+}} PT_P\right)=-b_1$.

Assume $m=1$. Since the sum $S_1(n)$ is nonzero if and only if $n=q-1$, taking $i=0$, we get $b_1\left(\sum_{P\in A_{1+}} T_P\right)=-b_1$.

2. Consider $(i_0,\dots,i_{d-1})$ as in the statement. By Corollary 5.10, we get that $b_1(\Theta_d(i_0,\dots,i_{d-1}))$ is
$$\sum_{\substack{\underline{n}\\ n_0+\ldots+n_d=m+q-2}} \binom{m+q-2}{\underline{n}} S_d(n_0+i_0,\dots,n_{d-1}+i_{d-1})a_{1+n_0+n_1q+\ldots+n_dq^d}.$$
We have $0\le n_j+i_j\le 2(q-1)$, hence we can evaluate $S_d(n_0+i_0,\dots,n_{d-1}+i_{d-1})$ thanks to Proposition 3.5. This sum is nonzero if and only if $n_j+i_j=q-1$ or $2(q-1)$ for all $j\in\{0,\dots,d-1\}$. If this happens, we have
$$d(q-1)\le\sum_{l=0}^{d-1}(n_l+i_l)\le i_0+\ldots+i_{d-1}+m+q-2$$
which contradicts $i_0+\ldots+i_{d-1}\le(d-1)(q-1)-m$. Accordingly, the sum always vanishes and $b_1(\Theta_d(i_0,\dots,i_{d-1}))=0$.
3. Apply the statement proved before to $i_0=l$ and $i_1=\ldots=i_{d-1}=0$.

$\square$

It is worth pointing out that the elements of $\mathbf{I}$ given in Theorem 1.1 are universal in the sense that, for a given type, they do not depend on the weight $k$ nor the ideal $\mathfrak{n}$. Some of them, as $\sum_{P\in A_{d+}} T_P$ for $d\ge 2$ for instance, are also independent of the type $m$. This means that, in the universal formal Hecke algebra $\mathbf{R}_A$, such an element is independent of $k$, $m$ and $\mathfrak{n}$.

***Remark 6.3***. — This phenomenon does not occur for classical modular forms of weight 2 as we now explain. Let $S_2(\Gamma_0(N))$ be the complex space of weight-2 cusp forms for $\Gamma_0(N)$ ($N\ge 1$). We write $(c_n)_{n\ge1}$ for the linear forms given by Fourier coefficients of such modular forms at the cusp infinity. The Hecke algebra $\mathbf{T}_c$ of weight 2 for $\Gamma_0(N)$ is the subring of $\mathrm{End}(S_2(\Gamma_0(N))$ spanned over $\mathbf{C}$ by all Hecke operators $T_n$ for $n\in\mathbf{N}$. Let $u_c$ be the $\mathbf{C}$-linear map $\mathbf{T}_c\to\mathrm{Hom}_{\mathbf{C}}(S_2(\Gamma_0(N),\mathbf{C})$ given by $s\mapsto c_1 s$. Relation (3) gives $c_n=u_c(T_n)$ for all $n\ge1$, thus $u_c$ is bijective. We claim that if there exists a $\mathbf{C}$-linear combination $s=\lambda_1T_{i_1}+\ldots+\lambda_jT_{i_j}$, with $j,\lambda_1,\dots,\lambda_j,i_1,\dots,i_j$ independent of $N$, such that $s=0$ as an endomorphism of $S_2(\Gamma_0(N))$, then the coefficients $\lambda_1,\dots,\lambda_s$ must be zero. In fact, when $N$ is prime, the Hecke operators $T_1,\dots,T_{g(N)}$ are $\mathbf{C}$-linearly independent in $\mathrm{End}(S_2(\Gamma_0(N))$ for $g(N)=\dim S_2(\Gamma_0(N))$ (this follows from the cusp infinity not being a Weierstrass point on the modular curve $X_0(N)$). Choosing $N$ prime such that $g(N)$ is large enough yields $\lambda_1=\ldots=\lambda_j=0$ and proves our claim.

In Section 7.2, we will further our investigation of the ideal $\mathbf{I}$ and prove that it vanishes in some cases (Theorem 7.7).

**6.2. Linear relations for eigenvalues. —**

***Notation 6.4.*** — Let $\mathfrak{p}$ an ideal of $A$ with monic generator $P$. A *Hecke eigenform* $f$ is a Drinfeld modular form which is an eigenform for all Hecke operators. We write $\lambda_P(f)$ for its eigenvalue for $T_P = T_{\mathfrak{p}}$.

For a Hecke eigenform $f$ such that $b_1(f) \neq 0$, Theorem 1.1 yields linear relations among its eigenvalues. It seems rather remarkable that these relations are universal in the sense that, for a fixed type, they do not depend on the weight $k$ nor on the level $\mathfrak{n}$.

***Proposition 6.5.*** — *Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ be a Hecke eigenform with $b_1(f) \neq 0$. If $m = 1$, we assume further $f \in M^2_{k,m}(\Gamma_0(\mathfrak{n}))$.*

1. *If $m \in \{0, 1\}$, then*
$$\sum_{P \in A_{1+}} P^{1-m} \lambda_P(f) + 1 = 0.$$
2. *Let $d \geq 1$ and $i_0, \dots, i_{d-1}$ satisfying* (1) *and* (2)*. Then*
$$\sum_{P \in A_{d+}} C(P)^{\underline{i}} \lambda_P(f) = 0.$$
3. *Let $l$ and $d$ be integers such that $0 \leq l \leq q - m$ and $d \geq (l + m)/(q - 1) + 1$. Then*
$$\sum_{P \in A_{d+}} P^l \lambda_P(f) = 0.$$
*In particular, if $d \geq 2$, or $f$ has type* 0 *and $d = 1$, then*
$$\sum_{P \in A_{d+}} \lambda_P(f) = 0.$$

**6.3. Linear relations for Hecke operators. —** We explain how some relations of Proposition 6.5 may follow from linear relations among Hecke operators in characteristic zero or $p$. In other words, we prove or suggest that certain elements of $\mathbf{I}$ given in Theorem 1.1 are zero in $\mathbf{T}'$.

***Notation 6.6.*** — For an ideal $\mathfrak{n}$ of $A$, let $\mathbf{H}_{\mathfrak{n}}$ be the abelian group of $\mathbf{Z}$-valued cuspidal harmonic cochains for $\Gamma_0(\mathfrak{n})$ on the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$ (we refer to Section 3 of [**15**] for the relevant definitions and properties). The group $\mathrm{GL}_2(K)$ acts on the left on the set of oriented edges $Y(\mathcal{T})$ of $\mathcal{T}$. We define an endomorphism $\theta_{\mathfrak{p}}$ of $\mathbf{H}_{\mathfrak{n}}$ by
$$(\theta_{\mathfrak{p}} F)(e) = \sum_{\substack{\alpha,\delta \text{ monic } \in A \\ \beta \in A,\, \deg \beta < \deg \delta \\ (\alpha\delta) = \mathfrak{p},\, (\alpha) + \mathfrak{n} = A}} F\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} e\right)$$
for $F \in \mathbf{H}_{\mathfrak{n}}$ and $e \in Y(\mathcal{T})$.

After scalar extension to the complex numbers $\mathbf{C}$, $\mathbf{H}_{\mathfrak{n}}$ is identified with a space of cuspidal automorphic forms on GL(2) over the adeles of $K$ (by the strong approximation theorem). Moreover, using Teitelbaum's residue map [**19**], Gekeler and Reversat [**15**] gave an isomorphism between $\mathbf{H}_{\mathfrak{n}}/p\mathbf{H}_{\mathfrak{n}}$ and a subspace of Drinfeld modular forms, namely the subspace $M^2_{2,1}(\Gamma_0(\mathfrak{n}), \mathbf{F}_p)$ of $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ consisting of such forms with residues in $\mathbf{F}_p$.

It turns out that this isomorphism is Hecke-equivariant, with the normalizations we have adopted here for $T_{\mathfrak{p}}$ and $\theta_{\mathfrak{p}}$. Finally, $M^2_{2,1}(\Gamma_0(\mathfrak{n}), \mathbf{F}_p)$ is an $\mathbf{F}_p$-vector space which, after scalar extension to $\mathbf{C}_\infty$, gives the whole space $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$. Put differently, the Hecke operator $T_{\mathfrak{p}}$ acting on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ can be thought of as the mod $p$ reduction of $\theta_{\mathfrak{p}}$.

***Lemma 6.7.*** — *Let $\mathfrak{n}$ be a prime. Assume $d \geq \deg(\mathfrak{n}) - 1$. Then $\sum_{\deg \mathfrak{p}=d} \theta_{\mathfrak{p}} = 0$. In particular, $\sum_{\deg \mathfrak{p}=d} T_{\mathfrak{p}} = 0$ on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$.*

*Proof.* — Let $F \in \mathbf{H}_{\mathfrak{n}}(\mathbf{C}) = \mathbf{H}_{\mathfrak{n}} \otimes_{\mathbf{Z}} \mathbf{C}$ be an eigenform for $(\theta_{\mathfrak{p}})_{\mathfrak{p}}$ with eigenvalues $(\lambda_{\mathfrak{p}})_{\mathfrak{p}}$. For $d > \deg(\mathfrak{n}) - 3$, we have

$$\sum_{\deg \mathfrak{p} \leq d} \lambda_{\mathfrak{p}} = 0. \tag{12}$$

It is essentially a consequence of the cuspidality of $F$. Namely, by the structure of the quotient graph $\Gamma_0(\mathfrak{n})\backslash\mathcal{T}$, the edges of $\mathcal{T}$ corresponding to matrices $\left(\begin{smallmatrix} \pi^k & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $k \geq \deg \mathfrak{n}$ are not in the support of $F$ (see also [**18**]). Using the Fourier expansion of $F$ and the relation between Fourier coefficients and Hecke eigenvalues $(\lambda_{\mathfrak{p}})_{\mathfrak{p}}$ ((3.12') and (3.13) in [**8**]), we derive (12). Since $\mathfrak{n}$ is prime, there exists a basis of $\mathbf{H}_{\mathfrak{n}}(\mathbf{C})$ consisting of normalized eigenforms for $(\theta_{\mathfrak{p}})_{\mathfrak{p}}$. Hence we have $\sum_{\deg \mathfrak{p} \leq d} \theta_{\mathfrak{p}} = 0$ if $d > \deg(\mathfrak{n}) - 3$. An equivalent formulation is: $\sum_{\deg \mathfrak{p} \leq \deg(\mathfrak{n})-2} \theta_{\mathfrak{p}} = 0$ and $\sum_{\deg \mathfrak{p}=d} \theta_{\mathfrak{p}} = 0$ if $d \geq \deg(\mathfrak{n}) - 1$. This completes the proof. □

Therefore, from the theory of automorphic forms, we know that certain elements of $\mathbf{I}$ given in Theorem 1.1 are zero on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$, because so they are on $\mathbf{H}_{\mathfrak{n}}$: this is the case for $\sum_{\deg \mathfrak{p}=d} T_{\mathfrak{p}}$ if $\mathfrak{n}$ is prime and $d \geq \deg(\mathfrak{n}) - 1$.

It is now natural to ask whether some elements of $\mathbf{I}$ in Theorem 1.1 can act nontrivially on $\mathbf{H}_{\mathfrak{n}}$ and be zero in $\mathbf{T}'$ (i.e. in characteristic $p$). We suggest that this happens.

***Question 6.8.*** — *Assume $\mathfrak{n}$ is prime. Do the following relations among Hecke operators on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$:*

1. $\sum_{\deg \mathfrak{p} \leq 1} T_{\mathfrak{p}} = 0$ *if $\mathfrak{n}$ has degree* $4$
2. $\sum_{\deg \mathfrak{p}=\deg(\mathfrak{n})-2} T_{\mathfrak{p}} = 0$ *if $\mathfrak{n}$ has degree* $\geq 4$

*hold?*

We checked numerically such relations on several examples. We computed Hecke operators on $\mathbf{H}_{\mathfrak{n}}/p\mathbf{H}_{\mathfrak{n}}$, for $\mathfrak{n}$ prime, using Teitelbaum's modular symbols for $\mathbf{F}_q(T)$ [**20**]. The first relation has been checked for $q \in \{2, 3, 4, 5, 7\}$ and the second one for all primes $\mathfrak{n}$ of degree 5 and 6 in $\mathbf{F}_2[T]$. Note that, when $\deg \mathfrak{n} = 4$, both relations are equivalent: indeed, we have $\sum_{\deg \mathfrak{p} \leq 2} \theta_{\mathfrak{p}} = 0$ (see proof of Lemma 6.7).

An affirmative answer to Question 6.8 would tell that some elements of $\mathbf{I}$ would be zero in $\mathbf{T}'$ but may be nonzero on the automorphic level, more precisely:

- $\sum_{\deg \mathfrak{p} \leq 1} T_{\mathfrak{p}} = 0$ in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$ for $\mathfrak{n}$ prime of degree 4;
- $\sum_{\deg \mathfrak{p}=\deg(\mathfrak{n})-2} T_{\mathfrak{p}} = 0$ in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$ for $\mathfrak{n}$ prime of degree $\geq 4$.

In the next paragraph, we are interested in the reverse problem: finding nonzero elements in the ideal $\mathbf{I}$.

**6.4. Nonzero elements in the annihilator.** — The following conjecture suggests that, in general, the Hecke annihilator $\mathbf{I}$ of $b_1$ is nonzero.

***Conjecture 6.9.*** — *Assume $\mathfrak{n}$ is prime of degree $\geq 5$. Then $\sum_{\deg \mathfrak{p} \leq 1} T_{\mathfrak{p}} \in \mathbf{I}$ is nonzero as an endomorphism of $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$. In particular, the map*

$$\begin{array}{rcl} u : \mathbf{T}' & \longrightarrow & \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{2,1}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty) \\ s & \longmapsto & b_1 s \end{array}$$

*is not surjective.*

The last statement follows from Lemma 6.2. As in Section 6.3, we were able to compute the action of $\sum_{\deg \mathfrak{p} \leq 1} T_{\mathfrak{p}}$ on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ on some examples. We checked Conjecture 6.9 for all primes $\mathfrak{n}$ in $\mathbf{F}_2[T]$ of degree in $\{5, 6, 7, 8, 9\}$, in $\mathbf{F}_3[T]$ of degree in $\{5, 6, 7, 8\}$, in $\mathbf{F}_4[T]$ and $\mathbf{F}_5[T]$ of degree 5 .

## 7. Proof and applications of Theorem 1.2

### 7.1. Explicit version of Theorem 1.2. —

***Notation 7.1.*** — We call a *decomposition of* $c \in \mathbf{N}$ a tuple $\underline{c} = (c_0, \ldots, c_d)$ such that $c = \sum_{j=0}^d c_j q^j$ and $0 \leq c_j < q$ for any $j \in \{0, \ldots, d\}$, for some $d \geq 0$. The *length of* $\underline{c}$ is $d+1$. Note that we do not require $c_d \neq 0$. The base $q$ expansion gives a decomposition of $c$. By putting zeros at the end of any decomposition of $c$, we obtain decompositions of larger length.

If $\underline{i} = (i_0, \ldots, i_d)$ is a decomposition of $i \geq 0$, let

$$l(i) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} \sum_{\substack{Q | P, Q \in A_{1+} \\ (Q) + \mathfrak{n} = A}} Q^{k-1} \quad \in A.$$

We prove Theorem 1.2 by establishing the following explicit version.

***Theorem 7.2.*** — *Assume $q$ is a prime.*

1. *Suppose $m = 0$. Let $n = c/(q-1) \in \mathscr{S}$. We fix a decomposition $(c_0, \ldots, c_d)$ of $c$ of length $d+1$ for some $d \geq 0$ (therefore $c_0 + \ldots + c_d = q-1$). Let*
$$t_{c_0,\ldots,c_d} = (-1)^d \binom{q-2}{c_0 - 1, c_1, \ldots, c_d}^{-1} \sum_{P \in A_{d+}} \left\{ {P \atop q^{d+1} - c} \right\} T_P \quad \in \mathbf{R}_A.$$
*Then, for any $k$ and $\mathfrak{n}$, we have $b_n = b_1 t_{c_0,\ldots,c_d}$ in the dual space of $M_{k,0}(\Gamma_0(\mathfrak{n}))$.*
2. *Suppose $m = 1$. Let $n = c/(q-1) \in \mathscr{S}$. We fix a decomposition $(c_0, \ldots, c_d)$ of $c$ of length $d+1$ for some $d \geq 0$ with $c_d \neq q-1$ (therefore $c_0 + \ldots + c_d = q-1$). Let*
$$t'_{c_0,\ldots,c_d} = (-1)^d \binom{q-1}{c_0, \ldots, c_d}^{-1} \sum_{P \in A_{d+}} \left\{ {P \atop q^{d+1} - 1 - c} \right\} T_P \quad \in \mathbf{R}_A.$$
*Then, for any $k$ and $\mathfrak{n}$, we have*
$$b_n = b_1 t'_{c_0,\ldots,c_d} + (-1)^{d+1} \binom{q-1}{c_0, \ldots, c_d}^{-1} l(q^{d+1} - 1 - c) a_1$$
*in the dual space of $M_{k,1}(\Gamma_0(\mathfrak{n}))$.*

3. *Assume $m = 1$. Let $d \geq 1$ and*
$$t_d = (-1)^d \sum_{P \in A_{d+}} \left( \left\{ {P \atop q^d-1} \right\} - \sum_{i=0}^{d-1} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \right) T_P \quad \in \mathbf{R}_A.$$
*Then, for any $k$ and $\mathfrak{n}$, we have*
$$b_{q^d} = b_1 t_d + (-1)^d \left( -l(q^d-1) + \sum_{i=0}^{d-1} l(q^d - 1 - (q-1)q^i) \right) a_1 \tag{13}$$
*in the dual space of $M_{k,1}(\Gamma_0(\mathfrak{n}))$.*

***Remark 7.3.*** — 1. Since $q$ is prime and $\sum_{j=0}^d c_j = q-1$, the multinomial coefficients $\binom{q-1}{c_0,\ldots,c_d}$ and $\binom{q-2}{c_0-1,c_1,\ldots,c_d}$ are nonzero in $\mathbf{F}_p$, by Lucas's theorem, hence invertible.

2. On doubly cuspidal forms, $a_1$ vanishes and the expressions of Theorem 7.2 simplify and provide Theorem 1.2. Moreover, since $b_1\mathbf{T}'$ is contained in the $\mathbf{C}_\infty$-vector space spanned by $b_n$ for $n \in \mathscr{S}$ (Corollary 5.8), we get the equality provided that $q$ is prime and $m \in \{0,1\}$.
3. For a given $n \in \mathscr{S}$, we get infinitely many expressions $s_n \in \mathbf{T}'$ such that $b_n = b_1 s_n$. The reason is that, in the first two items of Theorem 7.2, *any* decomposition of $c = (q-1)n$ gives rise to a formula for $s_n \in \mathbf{T}'$ satisfying the desired property. More generally, any element of $s_n + \mathbf{I}$ would satisfy the same property.
4. The primality assumption on $q$ is not always essential: it is required to ensure that the multinomial coefficient $\binom{q-1}{c_0,\ldots,c_d}$ for $m = 1$ (resp. $\binom{q-2}{c_0-1,c_1,\ldots,c_d}$ for $m = 0$) is nonzero in $\mathbf{F}_p$. Hence, the assumption is unnecessary in (13). If $q$ is not a prime, the first (resp. second) statement of Theorem 7.2 is true for $n = c/(q-1) \in \mathscr{S}$ such that there exists a decomposition $(c_0, \ldots, c_d)$ of $c$ with $\binom{q-2}{c_0-1,c_1,\ldots,c_d} \neq 0$ in $\mathbf{F}_p$ (resp. $\binom{q-1}{c_0,\ldots,c_d} \neq 0$ in $\mathbf{F}_p$) for some $d \geq 0$.

Before proving Theorem 7.2, we give an example.

***Example 7.4*** **($d = 1$).** — We put
$$s_n = -\binom{q-1}{n-1}^{-1} \sum_{P \in A_{1+}} P^{n-1} T_P \qquad \text{for } 1 \leq n \leq q-1$$
$$s_q = -\sum_{P \in A_{1+}} (P^{q-1} - 1) T_P.$$
Then $b_n(f) = b_1(s_n(f))$ for all $f \in M^2_{k,1}(\Gamma_0(\mathfrak{n}))$ and $1 \leq n \leq q$. This is valid for $q$ a power of a prime, by Remark 7.3 and Lucas's theorem. Using these formulas, we can recover the first $q$ coefficients of any Hecke eigenform $f$ in $M^2_{k,1}(\Gamma_0(\mathfrak{n}))$ in terms of $b_1(f)$ and the eigenvalues.

*Proof of Theorem 7.2.* — 1. Assume that the type $m$ is 0. We put $n_0 = c_0 - 1, n_1 = c_1, \ldots, n_d = c_d$, so that $n_0 + \ldots + n_d = q-2$. By Corollary 5.10, $a_{q-1}(\Theta_d(q-1-n_0, \ldots, q-1-n_{d-1}))$ is
$$\sum_{\underline{r}} \binom{q-2}{\underline{r}} S_d(r_0 + q - 1 - n_0, \ldots, r_{d-1} + q - 1 - n_{d-1}) a_{1+r_0+r_1q+\ldots+r_dq^d}$$

where $\underline{r} = (r_0, \ldots, r_d)$ satisfies $r_0 + \ldots + r_d = q-2$. From $n_i \geq -1$, we get $0 \leq r_i + q - 1 - n_i \leq 2(q-1)$ for all $i$. We can thus evaluate the sum $S_d(r_0 + q - 1 - n_0, \ldots, r_{d-1} + q - 1 - n_{d-1})$ by Proposition 3.5: it is nonzero only if $\underline{r}$ is such that $r_i \in \{n_i, q-1+n_i\}$, for all $i \in \{0, \ldots d-1\}$. Since $r_i \leq q-2$, we have $r_0 = n_0, \ldots, r_{d-1} = n_{d-1}$ and by Proposition 3.5,

$$a_{q-1}(\Theta_d(q-1-n_0, \ldots, q-1-n_{d-1})) = \tbinom{q-2}{\underline{n}}(-1)^d a_{1+n_0+n_1q+\ldots+n_dq^d}.$$

Finally, $a_{1+n_0+\ldots+n_dq^d} = a_{n(q-1)} = b_n$ and the conclusion follows.

2. Assume that the type $m$ is 1. Since $q^{d+1}-1-c$ has base $q$ expansion $\sum_{j=0}^{d}(q-1-c_j)q^j$, we have

$$\begin{aligned}\sum_{P \in A_{d+}} \{{}_{q^{d+1}-1-c}^{\ P}\} T_P &= \sum_{P \in A_{d+}} C_{P,0}^{q-1-c_0} \ldots C_{P,d-1}^{q-1-c_{d-1}} T_P \\ &= \Theta_d(q-1-c_0, \ldots, q-1-c_{d-1}).\end{aligned}$$

By Corollary 5.10, $b_1(\Theta_d(q-1-c_0, \ldots, q-1-c_{d-1}))$ is

$$\begin{aligned}&\sum_{\underline{r}} \tbinom{q-1}{\underline{r}} S_d(r_0+q-1-c_0, \ldots, r_{d-1}+q-1-c_{d-1}) a_{1+r_0+r_1q+\ldots+r_dq^d} \\ &+ l(q^{d+1}-1-c)a_1\end{aligned}$$

with $\underline{r} = (r_0, \ldots, r_d)$ such that $r_0+\ldots+r_d = q-1$. From $c_i \geq 0$ and $0 \leq r_i \leq q-1$, we get $0 \leq r_i+q-1-c_i \leq 2(q-1)$. Thus the sum $S_d(r_0+q-1-c_0, \ldots, r_{d-1}+q-1-c_{d-1})$ can be evaluated thanks to Proposition 3.5: it is nonzero if and only if $r_i \in \{c_i, q-1+c_i\}$ for all $i \in \{0, \ldots, d-1\}$.

Suppose there exists $k \in \{0, \ldots, d-1\}$ with $r_k = q-1+c_k$. Then, according to the previous remarks, we have

$$q-1-r_d = \sum_{j=0}^{d-1} r_j = q-1+c_k + \sum_{j=0, j\neq k}^{d-1} r_j \geq q-1+\sum_{j=0}^{d-1} c_j = 2(q-1)-c_d$$

hence $0 \leq q-1-c_d \leq -r_d$. This implies $r_d = 0$, thus $c_d = q-1$, which is impossible. Therefore, we have $r_j = c_j$ for any $j \in \{0, \ldots, d-1\}$ and $r_d = c_d$ as a consequence. Proposition 3.5 then provides

$$\begin{aligned}b_1(\Theta_d(q-1-c_0, \ldots, q-1-c_{d-1})) =& a_{1+c_0+c_1q+\ldots+c_dq^d} \\ &+ (-1)^d \tbinom{q-1}{c_0, \ldots, c_d}^{-1} l(q^{d+1}-1-c)a_1.\end{aligned}$$

Finally, $a_{1+c_0+c_1q+\ldots+c_dq^d} = a_{1+n(q-1)} = b_n$, thus the statement is proved.

3. Assume that the type $m$ is 1. We first compute $b_1(\Theta_d(q-1, \ldots, q-1))$. According to Corollary 5.10, it is

$$\begin{aligned}&\sum_{\substack{\underline{r}=(r_0,\ldots,r_d) \\ r_0+\ldots+r_d=q-1}} \tbinom{q-1}{\underline{r}} S_d(r_0+q-1, \ldots, r_{d-1}+q-1) a_{1+r_0+r_1q+\ldots+r_dq^d} \\ &+ l(q^{d+1}-1)a_1.\end{aligned}$$

By Proposition 3.5, the sum $S_d(r_0+q-1, \ldots, r_{d-1}+q-1)$ is nonzero if and only if $r_i \in \{0, q-1\}$ for all $i \in \{0, \ldots, d-1\}$. This means that $(r_0, \ldots, r_{d-1})$ is one of the following:

$$(q-1,0,\dots,0)\ ,(0,q-1,0,\dots,0)\ ,\dots\ ,(0,\dots,0,q-1)\ ,(0,\dots,0).$$

Thus $b_1(\Theta_d(q-1,\dots,q-1))$ equals

$$(-1)^d\left(a_{1+(q-1)}+\ldots+a_{1+(q-1)q^{d-1}}+a_{1+(q-1)q^d}\right)+l(q^{d+1}-1)a_1. \tag{14}$$

Next, we compute $b_1(\Theta(q-1,\dots,0,\dots,q-1))$, the only zero term being at the $(j+1)$th position $(0\le j\le d-1)$. From Corollary 5.10, it is

$$\sum_{\substack{\underline{r}=(r_0,\dots,r_d)\\ r_0+\ldots+r_d=q-1}} \tbinom{q-1}{\underline{r}} S_d(r_0+q-1,\dots,r_j,\dots,r_{d-1}+q-1)a_{1+r_0+r_1q+\ldots+r_dq^d}$$
$$+\,l(q^{d+1}-1-(q-1)q^j)a_1.$$

Again by Proposition 3.5, the sum is only over $\underline{r}$ satisfying the following two properties:

$$\begin{aligned} &r_i\in\{0,q-1\} \quad \text{for all } i\in\{0,\dots,d-1\}, i\neq j\\ &r_j\in\{q-1,2(q-1)\}.\end{aligned}$$

Since $r_0+\ldots+r_d=q-1$, we have necessarily $r_j=q-1$, $r_i=0$ for all $i\neq j$ and $r_d=0$. Then

$$b_1(\Theta(q-1,\dots,0,\dots,q-1))=(-1)^d a_{1+(q-1)q^j}+l(q^{d+1}-1-(q-1)q^j)a_1 \tag{15}$$

Combining (14) and (15), we get the claim.

□

**7.2. Applications.** — Theorem 1.2 has the following straightforward consequence.

***Corollary 7.5.*** — *Under the assumptions of Theorem 1.2, if $f$ is a Hecke eigenform with $b_n(f)\neq 0$ for some $n\in\mathscr{S}$, then $b_1(f)\neq 0$.*

In particular, in Proposition 6.5, one can replace the assumption $b_1(f)\neq 0$ by: there exists $n\in\mathscr{S}$ such that $b_n(f)\neq 0$.

We now provide multiplicity one statements in certain spaces of Drinfeld modular forms.

***Lemma 7.6.*** — 1. *Let $d=\dim M_{k,m}(\mathrm{GL}_2(A))$. The $\mathbf{C}_\infty$-linear map*

$$\begin{array}{rcl} M_{k,m}(\mathrm{GL}_2(A)) & \longrightarrow & \mathbf{C}_\infty^d\\ f & \longmapsto & (b_0(f),\dots,b_{d-1}(f))\end{array}$$

*is an isomorphism.*

2. *Let $d=\dim M^2_{2,1}(\Gamma_0(\mathfrak{n}))$. The $\mathbf{C}_\infty$-linear map*

$$\begin{array}{rcl} M^2_{2,1}(\Gamma_0(\mathfrak{n})) & \longrightarrow & \mathbf{C}_\infty^d\\ f & \longmapsto & (b_1(f),\dots,b_d(f))\end{array}$$

*is an isomorphism.*

*Proof.* — The first assertion follows readily from a formula relating, for a nonzero $f \in M_{k,m}(\mathrm{GL}_2(A))$, the orders of vanishing of $f$ at elliptic, non-elliptic points and the cusp infinity of $\mathrm{GL}_2(A)$ (see (5.14) in Gekeler's paper [**7**]).

For the second assertion, we consider the Drinfeld modular curve $X_0(\mathfrak{n})$ attached to $\Gamma_0(\mathfrak{n})$. This smooth projective algebraic curve over $\mathbf{C}_\infty$ is the compactification of the affine curve $Y_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n})\backslash\Omega$ over $\mathbf{C}_\infty$, the group $\Gamma_0(\mathfrak{n})$ acting on $\Omega$ via linear fractional transformations. Actually, the curve $Y_0(\mathfrak{n})$ is a coarse moduli scheme for rank-2 Drinfeld modules with a level structure determined by $\mathfrak{n}$. The cusps, i.e. the set $X_0(\mathfrak{n}) - Y_0(\mathfrak{n})$, is naturally in bijection with $\Gamma_0(\mathfrak{n})\backslash\mathbf{P}^1(K)$. Since we assume $\mathfrak{n}$ prime, the cusps are labeled $\{0, \infty\}$ as usual. Gekeler gave formulas for the genus $g = g(X_0(\mathfrak{n}))$ in terms of the degree of $\mathfrak{n}$ ([**5, 6**]).

One can show that $\infty$ is not a Weierstrass point on $X_0(\mathfrak{n})$ with $\mathfrak{n}$ prime (i.e. every holomorphic differential form on $X_0(\mathfrak{n})$ vanishes at $\infty$ at order $< g$). This is merely an adaptation of Ogg's geometric argument [**16**] for the classical modular curve $X_0(pM)$ with $p$ prime and $p \nmid M$. To adapt the proof, we need Gekeler's description of the bad fiber at $\mathfrak{n}$ of a model of $X_0(\mathfrak{n})$ over $A$ ([**6**] p. 233): it consists of two copies of the projective line over $\mathbf{F}_\mathfrak{n} = A/\mathfrak{n}$ intersecting transversally at $n$ supersingular points (i.e. points whose underlying rank-2 Drinfeld module is supersingular over $\mathbf{F}_\mathfrak{n}$). The full Atkin–Lehner involution $w_\mathfrak{n}$ interchanges the components. The reductions of the cusps are on distinct components hence are not supersingular points. The second ingredient is the analogue of Ogg's formula (2) in [**16**]: the number $n$ of supersingular points is $1 + g$, according to (5.4) in [**6**] and Gekeler's formula for $g$. We leave the details to the reader.

The map $f \mapsto f(z)dz$ defines an isomorphism between $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ and the space of holomorphic differential forms on $X_0(\mathfrak{n})$ ([**15**] Proposition 2.10.2), hence both spaces have dimension $g = d$. Since $\infty$ is not a Weierstrass point, any Drinfeld modular form in $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ vanishes at $\infty$ at order $< d$. In other words, the linear map $M^2_{2,1}(\Gamma_0(\mathfrak{n})) \to \mathbf{C}^d_\infty$ given by $f \mapsto (b_1(f), \ldots, b_d(f))$ is injective, hence bijective. □

***Theorem 7.7.*** — *Let $M$ be one of the following spaces of Drinfeld modular forms:*

1. $M^1_{k,0}(\mathrm{GL}_2(A))$ *with* $k < (q+1)^2(q-1)$
2. $M^2_{k,1}(\mathrm{GL}_2(A))$ *with* $k < q^2(q+1)$
3. $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ *with* $\mathfrak{n}$ *prime of degree* 3.

*Then:*

- *Any eigenform in $M$ for the operators $(T_\mathfrak{p})_{\deg \mathfrak{p}=1}$ is characterized in the space $M$ by its eigenvalues, up to a multiplicative constant.*
- *The map $u : \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M, \mathbf{C}_\infty)$ is an isomorphism.*

*Proof.* — Consider the first two cases for $M$. By the cuspidality (resp. doubly cuspidality) condition and the assumption on the type, we have $b_0(f) = a_m(f) = 0$. Therefore, any function $f \in M$ is determined, in the space $M$, by its coefficients $b_1(f), \ldots, b_{d-1}(f)$, according to Lemma 7.6. Now, if $f$ is an eigenform for $(T_\mathfrak{p})_{\deg \mathfrak{p}=1}$, we know that $b_1(f), \ldots, b_q(f)$ are determined by the eigenvalues (up to a multiplicative constant), thanks to Example 7.4. Recall that the dimension of $M_{k,m}(\mathrm{GL}_2(A))$ is $d = \lfloor (k - (q+1)m)/(q^2-1) \rfloor + 1$ (this follows from Gekeler's formula (5.14) in [**7**]). Here, the assumptions on the weight $k$ ensure that $d - 1 \le q$. The conclusion follows.

The proof of the third case is similar, except that the dimension of $M$ is $q$. Indeed, this dimension is equal to the genus of $X_0(\mathfrak{n})$. By Gekeler's formula for the genus ([**5, 6**]), it is $q$ when $\mathfrak{n}$ is prime of degree 3.

For the bijectivity of $u$, we need only to prove the surjectivity by Lemma 6.2. Consider the first two cases for $M$. As before, $M$ has dimension $d-1 \leq q$. Moreover, the image of $u$ contains $b_1, \dots, b_{d-1}$ (by Theorem 7.2) which are linearly independent (by Lemma 7.6), hence the conclusion. The proof of the third case is similar. □

As a corollary, we get that the dimension of the $\mathbf{C}_\infty$-algebra $\mathbf{T}'$ coincides with the dimension of the space of Drinfeld modular forms $M$, for $M$ as in the statement.

**7.3. Comment on $A$-structures.** — Although we worked with $\mathbf{C}_\infty$-structures, most of the results of this paper could be transferred over the ring $A$. For instance, one could work with the subspace $M^2_{k,m}(\Gamma_0(\mathfrak{n}); A) \subset M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ consisting of modular forms with expansion in $A[[t]]$ and the Hecke algebra $\mathbf{T}'_A$ spanned over $A$ by Hecke operators. Using Proposition 5.2, one may check that the map

$$\mathbf{T}'_A \to \mathrm{Hom}_A(M^2_{k,m}(\Gamma_0(\mathfrak{n}); A), A)$$

induced by $s \mapsto b_1 s$, is well-defined. We expect that $M^2_{k,m}(\Gamma_0(\mathfrak{n}); A)$ is a $A$-structure of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ (i.e. there exists a basis of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ consisting of modular forms with coefficients in $A$). However, a general theory of such algebraic Drinfeld modular forms is still missing in the literature. Some instances of such a theory can be found in [**12**] (Section 2, for $M_{k,m}(\mathrm{GL}_2(A))$) and [**1**] (Section 4.2, for $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$).

## 8. Coefficients of $h$

We use Theorem 7.2 to compute explicitly some coefficients of Gekeler's Drinfeld modular form $h$, defined in [**7**]. Recall that $h$ has weight $q+1$ and type 1 for $\mathrm{GL}_2(A)$. It is defined as a certain Poincaré series and is also a $(q-1)$th root of the Drinfeld discriminant form $\Delta$. Moreover, it is a cuspidal Hecke eigenform with $T_\mathfrak{p} h = h$ for any $\mathfrak{p}$ (Corollary 7.6 in [**7**] with a different normalization of Hecke operators). The first coefficients of $h$ are $a_1(h) = -1$ and $b_1(h) = a_q(h) = \begin{cases} 0 & \text{if } q > 2 \\ 1 & \text{if } q = 2 \end{cases}$.

***Proposition 8.1.*** — *For $P$ in $A$, let $\sigma_P = \sum_{Q|P, Q\in A_{1+}} Q^q$.*

1. *Assume $q$ is a prime $> 2$. Let $c \in \mathbf{N}$ such that $c = \sum_{j=0}^d c_j q^j$ with $0 \leq c_j < q$, $\sum_{j=0}^d c_j = q-1$ and $c_d \neq q-1$ (we do not necessarily assume $c_d \neq 0$). Then*

$$b_{\frac{c}{q-1}}(h) = (-1)^d \binom{q-1}{c_0, \dots, c_d}^{-1} \sum_{P\in A_{d+}} \left\{ {P \atop q^{d+1}-1-c} \right\} \sigma_P. \tag{16}$$

*Moreover, for $d \geq 0$,*

$$b_{q^d}(h) = (-1)^{d+1} \sum_{P\in A_{d+}} \left( -\left\{ {P \atop q^d-1} \right\} + \sum_{i=0}^{d-1} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \right) \sigma_P \tag{17}$$

2. *Assume $q=2$. Then for every $d \geq 0$, one has*

$$b_{2^d}(h) = (-1)^d \sum_{P \in A_{d+}} \left( -\genfrac\{\}{0pt}{}{P}{2^d-1} + \sum_{i=0}^{d-1} \genfrac\{\}{0pt}{}{P}{2^d-1-2^i} \right) (1+\sigma_P). \tag{18}$$

***Remark 8.2.*** — We recover that the corresponding coefficients of $h$ are polynomials in $T^q - T$ with coefficients in $\mathbf{F}_q$ (indeed, they are elements of $A$ which are invariant under $T \mapsto T + c$ for $c \in \mathbf{F}_q$). More generally, Gekeler proved that this property holds for any coefficient of $h$ (Theorem 2.4 of [**9**]).

Taking $d = 1$ in Proposition 8.1, one can recover the first $q$ coefficients of $h$. If $q$ is a prime $> 2$, then $b_i(h) = 0$ if $1 \leq i \leq q-2$, $b_{q-1}(h) = -1$ and $b_q(h) = T^q - T$. They can also be obtained from the Taylor series $h = -tU_1^{-1} + o(t^{1+(q-1)(q^3-q^2)})$ with $U_1 = 1 - t^{(q-1)^2} + (T^q - T)t^{(q-1)q}$ (see Corollary 10.4 in [**7**]).

For $i \in \mathbf{N}$, let $[i] = T^{q^i} - T$. Using congruences and estimates on the degree of coefficients of $h$, Gekeler proved that for any $d \geq 1$,

$$b_{q^d}(h) = \begin{cases} [d] & \text{if } q > 2 \\ 1 + [d] & \text{if } q = 2 \end{cases} \tag{19}$$

(see Corollary 2.6 of [**9**]; note that his $b_i$ denotes our $-b_i$). Equation (17) thus provides an alternative formula for $b_{q^d}(h)$. We have not been able to recover (19) from (17) and (18). Hence we derive some arithmetic identities in $\mathbf{F}_q[T]$ which may be nontrivial and of some interest.

***Corollary 8.3.*** — *Let $q$ be a prime $> 2$ and $d \geq 1$.*

1.
$$[d] = (-1)^{d+1} \sum_{P \in A_{d+}} \left( -\genfrac\{\}{0pt}{}{P}{q^d-1} + \sum_{i=0}^{d-1} \genfrac\{\}{0pt}{}{P}{q^d-1-(q-1)q^i} \right) \sigma_P.$$

2. *For $0 \leq i \leq d-1$,*
$$(-1)^d [i] = \sum_{P \in A_{d+}} \genfrac\{\}{0pt}{}{P}{q^d-1-(q-1)q^i} \sigma_P.$$

3.
$$(-1)^d \sum_{i=1}^{d} [i] = \sum_{P \in A_{d+}} \genfrac\{\}{0pt}{}{P}{q^d-1} \sigma_P.$$

*Proof.* — The first one follows from (17) and (19). For the second one, we first apply (16) to $c = (q-1)q^i$ with $0 \leq i \leq d-1$ and get

$$(-1)^d b_{q^i}(h) = \sum_{P \in A_{d+}} \genfrac\{\}{0pt}{}{P}{q^{d+1}-1-(q-1)q^i} \sigma_P = \sum_{P \in A_{d+}} \genfrac\{\}{0pt}{}{P}{q^d-1-(q-1)q^i} \sigma_P$$

where the last equality follows from $q^{d+1} - 1 - (q-1)q^i = (q-1)q^d + \sum_{j=0}^{d-1}(q-1)q^j - (q-1)q^i$ and $\deg P = d$. With (19), we get the second claim. The third one is obtained by combining the first two identities. □

TABLE 1. $q = 3$, $d \leq 4$

| $i$ | $b_i(h)$ |
|---|---|
| 1 | $0$ |
| 2 | $-1$ |
| 3 | $[1]$ |
| 5 | $-[1]$ |
| 6 | $-[1]^2 - 1$ |
| 9 | $[2] = [1]^3 + [1]$ |
| 14 | $[1]^4 - 1$ |
| 15 | $[1]^5 - [1]^3 + [1]$ |
| 18 | $-[1]^6 + [1]^4 - [1]^2 - 1$ |
| 27 | $[3] = [1]^9 + [1]^3 + [1]$ |
| 41 | $-[1]^{13} + [1]^9 - [1]^7 - [1]$ |
| 42 | $-[1]^{14} + [1]^{12} - [1]^{10} - [1]^8 - [1]^2 - 1$ |
| 45 | $[1]^{15} - [1]^{13} + [1]^{11} - [1]^9 + [1]^3 + [1]$ |
| 54 | $-[1]^{18} + [1]^{12} + [1]^{10} - [1]^6 + [1]^4 - [1]^2 - 1$ |
| 81 | $[4] = [1]^{27} + [1]^9 + [1]^3 + [1]$ |

In Table 1, we provide further examples of coefficients of $h$ from Proposition 8.1. Observe that when $i$ is even (resp. odd), $b_i(h)$ is an even (resp. odd) polynomial in $[1] = T^q - T$. This is more generally true for any coefficient when $q = 3$: it follows from the coefficients of $h$ being balanced, a property established by Gekeler (Theorem 2.4 of [**9**]). Note that, in our table, the constant term is $-1$ when $i$ is even: we wonder if such a statement holds more generally.

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